MAX-PLANCK-INSTITUT FÜR WISSENSCHAFTSGESCHICHTE
Max Planck Institute for the History of Science

2016

PREPRINT 480

Yvonne Fourès-Bruhat

Existence theorem for certain systems of nonlinear partial differential equations

Existence theorem for certain systems of nonlinear partial differential equations

Yvonne Fourès-Bruhat*

Abstract

This paper is the translation by Giampiero Esposito of a paper originally published in French in *Acta Mathematica* **88**, 141-225 (1952) under the title: **Théorème d'existence for certain systèmes d'équations aux dérivées partielles non linéaires**. The first three chapters are devoted to the solution of the Cauchy problem, in the nonanalytic case, for a system of nonlinear second-order hyperbolic partial differential equations with n unknown functions and four independent variables. This task is accomplished in chapter III by using the system of integral equations fulfilled by the solutions of partial differential equations that approximate the original nonlinear system. In chapter IV, such results are applied to the vacuum Einstein equations. The resulting Ricci-flatness condition is expressed, in isothermal coordinates, through nonlinear equations of the kind studied here. It is hence proved that the solution of the Cauchy problem, pertaining to such nonlinear equations, satisfies over the whole of its existence domain the isothermal conditions if the same is true for the initial data. One therefore obtains a solution of the vacuum Einstein equations which is unique up to a coordinate change.

Introduction

I have studied the formulation of the Cauchy problem for nonlinear hyperbolic partial differential equations as suggested by the equations of Einstein's gravitation. These equations are indeed a system of ten second-order equations, with four independent variables (space and time) and ten unknown functions, the gravitational potentials. These equations are of normal hyperbolique type in a regular system of spacetime coordinates. The determinism problem occurs, in Einstein's theory, in the form of a Cauchy problem, the data being assigned on a space-oriented manifold, with respect to this system of equations. The study of this problem, on assuming analytic Cauchy data[1], had shown that, by using four conditions fulfilled by these data, in correspondence to some initial data, assigned on a noncharacteristic surface S, there existed an Einstein spacetime in the neighbourhood of S. The study of characteristic surfaces, defined by the fact that Cauchy data, assigned on such surfaces, do not determine in its neighbourhood a spacetime, had shown that these surfaces were tangent at any point M whatsoever on them to the characteristic conoid with vertex at M, this conoid being generated by light rays, i.e., null geodesics. One could also see the emergence of gravitational waves and gravitational rays, giving to the gravitational field the character of a propagation phenomenon, and one could see the identity of propagation laws for light and for the gravitational field. It therefore seemed very important to extend these results to nonanalytic Cauchy data, on the one hand because such an hypothesis of analyticity is meaningless in a physical theory where coordinate changes are only restricted to be sufficiently differentiable, on the other hand to highlight what M. Stellmacher [11] calls *causal structure* of spacetime: the gravitational field at a point M should only depend on the field at points preceding M (i.e. one can reach the point M by a future-directed timelike worldline, which has a lower bound for the time coordinate). M. Stellmacher had proved, by using majorizations of Friedrichs and Lewy type and isothermal coordinates, an uniqueness theorem: to Cauchy data assigned on a domain of a spacelike surface located within the characteristic conoid of vetex M, there corresponds at most one single system of potentials at the point M (up to a coordinate change). It has been our aim to prove that there corresponds effectively one such a system of gravitational potentials.

*Email: ycb@ihes.fr

[1] G. DARMOIS [1], A. LICHNEROWICZ [2].

The problem that I have considered is the Cauchy problem with respect to a system of second-order partial differential equations, which are only linear with respect to second derivatives. The universe being described by a system of isothermal and regular spacetime coordinates, the coefficients of second derivatives are the same for the ten equations, the corresponding quadratic form having to be of the normal hyperbolic type.

The solution of the Cauchy problem for a nonlinear hyperbolic partial differential equation has been determined by H. Lewy [5], in the case of two variables, by integration along characteristics and subsequent approximations. Schauder [7], by using majorizations of the Friedrichs and Lewy type and the approximation by means of analytic functions, pointed out in 1935 to a method enabling, no doubt, to obtain an existence theorem for a second-order equation, hyperbolic, in any number of independent variables. In 1937, by using a majorization discovered by Haar, Schauder [8] proved the existence of a solution of the Cauchy problem for certain systems of first-order equations. His solution was applicable, in particular, to a second-order equation in two variables. The study of first-order hyperbolic systems and the Fourier transform led on the other hand Petrovsky [9], after Herglotz [8], to a formulation of very general existence theorems.

It seemed to me that, for the problems considered by the theory of relativity, it would be interesting to obtain, under the minimal possible amount of assumptions, an existence theorem easy to use, enabling to find properties of the solutions that can be compared with the classical properties of light waves and gravitational potentials, and to have formulas which can be an efficient method of calculating gravitational fields, at least approximately, that correspond to given initial conditions.

I therefore devote the first three chapters of this work to the solution of the Cauchy problem, in the nonanalytic case, for a system of nonlinear second-order hyperbolic partial differential equations with n unknown functions W_s and four independent variables x^α, having the form

$$E_s = A^{\lambda\mu}\frac{\partial^2 W_s}{\partial x^\lambda \partial x^\mu} + f_s = 0,\ \lambda,\mu = 1,2,3,4,\ s = 1,2,...,n,$$

where $A^{\lambda\mu}$ and f_s are given functions of the unknown W_s and their first derivatives.

I use, for this solution, a system of integral equations fulfilled by the seven-times differentiable solutions of equations E.[2] This system is obtained for some linear equations by integrating over the characteristic conoid Σ with vertex M some linear combinations of the E equations (the coefficients of these combinations are some auxiliary functions which possess at M the parametrix properties, an approximation of the elementary solution of M. Hadamard) and by adjoining to the formulas of Kirchhoff type obtained in such a way the equations determining the characteristic conoid and the auxiliary functions. The results admit of an easy extension to nonlinear equations, subject to the condition of integrating over Σ not the E equations themselves but the equations derived from E by five derivatives, and provided one supplements the previous integral equations with the equations relating among themselves the derivatives of the unknown functions up to the fifth order. Such a system had been formed by Sobolev [11] for an hyperbolic linear second-order partial differential equation (with analytic coefficients) and by Christianovich [13] for a nonlinear equation in four variables. Christianovich was limiting himself, however, to an equation not containing mixed second derivatives, and was not writing Kirchhoff formulas except when assigning particular values to the coefficients (the solution that he provides of the system he obtains is on the other hand erroneous, the integrals that he considers not being convergent).

By extending these methods, I write in its complete form the system of integral equations satisfied from a system whatsoever of type E and I study in detail the various quantities occurring in these integral equations (chapters I and II) in light of the goal of solving them. I point out that the kernel, occurring in the Kirchhoff formula, is only bounded under differentiability assumptions made on the unknown functions. Some difficulties occur therefore in the process of solving directly the system of integral equations obtained, and of using it to solve the Cauchy problem relative to E.

[2] M. M. RIESZ uses equally well integral equations to solve the linear Cauchy problem with variable coefficients.

I come in chapter III to the Cauchy problem for system E by using the system of integral equations fulfilled by the solutions of partial differential equations E_1 that approximate E. The proof is performed in detail in the case, a bit simpler, that involves derivatives of a lower order, which is the one of equations of relativity, where the coefficients of second derivatives depend on the unknown functions but not on their first derivatives. I prove that, to Cauchy data five times differentiable, assigned on a compact domain d of the initial surface $x^4 = 0$, there corresponds a unique solution, four times differentiable, of equations E in a domain D, section of a cone having as basis the domain d, if the coefficients of these equations are four times differentiable.

The solution of the Cauchy problem for a system E whatsoever can be obtained in a completely analogous way: it is enough to consider equations approaching not E itself but some equations previously derived.

I apply, in chapter IV, the previous results to the equations of gravitation.

The equations of relativity $R_{\alpha\beta} = 0$ get reduced, in isothermal coordinates, to equations of the type E, $G_{\alpha\beta} = 0$. I prove, by using the conservation equations, that the solution of the Cauchy problem, pertaining to the equations $G_{\alpha\beta} = 0$, satisfies over the whole of its existence domain the isothermal conditions if the same is true for the initial data. This solution satisfies therefore the equations of gravitation. I prove that it is unique up to a coordinate change. I have also built an Einstein spacetime corresponding to nonanalytic initial data, assigned on a spacelike domain, and in such a way that it highlights the propagation character which is peculiar of relativistic gravitation.

CHAPTER I

Linear equations

We will consider in this chapter a system (E) of n second-order partial differential equations, with n unknown functions u_s and four variables x, hyperbolic and linear, of the following type:

$$E_r = A^{\lambda\mu} \frac{\partial^2 u_r}{\partial x^\lambda \partial x^\mu} + B_r^{s\ \mu} \frac{\partial u_s}{\partial x^\mu} + f_r = 0,\ r, s = 1, 2, ..., n,\ \lambda, \mu = 1, 2, ..., 4.$$

The coefficients $A^{\lambda\mu}$ (which are the same for all n equations), $B_r^{s\ \mu}$ and f_r are given functions of the four variables x^α. We will assume that they satisfy, within a domain defined by

$$\left|x^i - \overline{x}^i\right| \leq d,\ \left|x^4\right| \leq \varepsilon\ (i = 1, 2, 3)$$

(where $\overline{x}^i, d$ and ε are some given numbers) the following assumptions:

Assumptions on the coefficients

(1) The coefficients $A^{\lambda\mu}$ and $B_s^{r\ \lambda}$ possess continuous and bounded derivatives up to the orders *four* and *two*, respectively. The coefficients f_r are continuous and bounded.

(2) The quadratic form $A^{\lambda\mu} x_\lambda x_\mu$ is of the normal hyperbolic type, has one positive square and three negative squares. We will assume in addition that the variable x^4 is a temporal variable, the three variables x^i being spatial, i.e.

$$A^{44} > 0 \text{ and the quadratic form } A^{ij} x_i x_j < 0 \text{ (negative definite).}$$

(3) Partial derivatives of the $A^{\lambda\mu}$ of order four and two, respectively, and $B_s^{r\ \lambda}$ satisfy Lipschitz conditions with respect to all their arguments.

Summary of chapter I

We will prove, in light of our aim to solve the Cauchy problem, that every system of n functions (continuous and bounded within D with their first partial derivatives), satisfying the (E) equations and taking at $x^4 = 0$, as for their first partial derivatives, some given values, is a solution of a system of integral equations (I). These equations (I) express the values, at a point $M_0(x_0)$ belonging to D, of the unknown u_s in terms of their values on the characteristic chonoid (Σ_0) of vertex M_0 and in terms of the initial data.

We will obtain these equations by integrating over Σ_0 some linear combinations of equations (E), the coefficients of these combinations being n^2 auxiliary functions which exhibit a singularity at M_0.

We will assume, in part I of this chapter, that the coefficients $A^{\lambda\mu}$ take at M_0 some particular values $(1, 0$ and $-1)$. We will suppress this restriction in part II.

A. Characteristic conoid

1 Equations defining the characteristic conoid

The characteristic surfaces of system (E) are three-dimensional manifolds of the space of four variables x^α, solutions of the differential system

$$F = A^{\lambda\mu} y_\lambda y_\mu = 0$$

with

$$y_\lambda dx^\lambda = 0.$$

The four quantities y_λ denote a system of directional parameters of the normal to the contact element, having support x^α. Let us take this system, which is only defined up to a proportionality factor, in such a way that $y_4 = 1$ and let us set $y_i = p_i$. The desired surfaces are solution of

$$F = A^{44} + 2A^{i4} p_i + A^{ij} p_i p_j = 0,\ dx^4 + p_i dx^i = 0. \tag{1.1}$$

The characteristics of this differential system, bicharacteristics of equations (E), satisfy the following differential equations:

$$\frac{dx^i}{A^{i4} + A^{ij} p_j} = \frac{dx^4}{A^{44} + A^{i4} p_i} = \frac{-dp_i}{\frac{1}{2}\left(\frac{\partial F}{\partial x^i} - p_i \frac{\partial F}{\partial x^4}\right)} = d\lambda_1,$$

λ_1 being an auxiliary parameter.

The characteristic conoid Σ_0 with vertex $M_0(x_0^\alpha)$ is the characteristic surface generated from the bicharacteristics passing through M_0. Any such bicharacteristic satisfies the system of integral equations

$$\begin{gathered} x^i = x_0^i + \int_0^{\lambda_1} T^i d\lambda_1,\ T^i = A^{i4} + A^{ij} p_j, \\ x^4 = x_0^4 + \int_0^{\lambda_1} T^4 d\lambda_1,\ T^4 = A^{44} + A^{i4} p_i, \\ p_i = p_i^0 + \int_0^{\lambda_1} R_i d\lambda_1,\ R_i = -\frac{1}{2}\left(\frac{\partial F}{\partial x^i} - p_i \frac{\partial F}{\partial x^4}\right), \end{gathered} \tag{1.2}$$

where the p_i^0 verify the relation

$$A_0^{44} + 2A_0^{i4} p_i^0 + A_0^{ij} p_i^0 p_j^0 = 0, \tag{1.3}$$

where $A_0^{\lambda\mu}$ denotes the value of the coefficient $A^{\lambda\mu}$ at the vertex M_0 of the conoid Σ_0.

We will assume that at the point M_0 the coefficients $A^{\lambda\mu}$ take the following values:

$$A_0^{44} = 1,\ A_0^{i4} = 0,\ A_0^{ij} = -\delta^{ij}. \tag{1.4}$$

The relation (1.3) takes therefore the simple form

$$\sum \Big(p_i^0)^2 = 1.$$

We will introduce to define the points of the surface Σ_0, besides the parameter λ_1 which defines the position of a point on a given bicharacteristic, two new parameters λ_2 and λ_3 that vary with the bicharacteristic under consideration, by setting[3]

$$p_1^0 = \sin\lambda_2 \cdot \cos\lambda_3,\ p_2^0 = \sin\lambda_2 \cdot \sin\lambda_3,\ p_3^0 = \cos\lambda_2.$$

2 Domain V

The assumptions made on the coefficients $A^{\lambda\mu}$ make it possible to prove that there exists a number ε_1 defining a variation domain Λ of the parameters λ_i by means of

$$|\lambda_1| \leq \varepsilon_1,\ 0 \leq \lambda_2 \leq \pi,\ 0 \leq \lambda_3 \leq 2\pi, \tag{Λ}$$

such that the integral equations (1.2) possess within (Λ) a unique solution, continuous and bounded

$$x^\alpha = x^\alpha(x_0^\alpha, \lambda_1, \lambda_2, \lambda_3),\ p_i = p_i(x_0^\alpha, \lambda_1, \lambda_2, \lambda_3), \tag{2.1}$$

satisfying the inequalities

$$\left|x^i - \overline{x}^i\right| \leq d,\ \left|x^4\right| \leq \varepsilon$$

and possessing partial derivatives, continuous and bounded, of the first three orders with respect to the overabundant variables λ_1, p_i^0 (hence with respect to the three variables λ_i).

The first four equations (2.1) define, as a function of the three parameters λ_i, varying within the domain Λ, a point of a domain V of the characteristic conoid Σ_0.

We shall be led, in the following part of this work, to consider other parametric representations of the domain V:

(1) We shall take as independent parameters the three quantities $x^4, \lambda_2, \lambda_3$. The function $x^4(\lambda_1, \lambda_2, \lambda_3)$ satisfies the equation

$$x^4 = \int_0^{\lambda_1} T^4 d\lambda_1 + x_0^4 \text{ where } T^4 = A^{44} + A^{i4}p_i. \tag{2.2}$$

Or it turns out from (1.3) that, on Σ_0, one has

$$2A^{i4}p_i = -A^{ij}p_ip_j - A^{44} \geq -A^{44},$$

from which

$$T^4 \geq \frac{A^{44}}{2} > 0;$$

x^4 is thus a monotonic increasing function of λ_1, the correspondence between $(x^4, \lambda_2, \lambda_3)$ and $(\lambda_1, \lambda_2, \lambda_3)$ is bijective.

(2) We shall take as representative parameters of a point of Σ_0 his three spatial coordinates x^i. The elimination of $\lambda_1, \lambda_2, \lambda_3$ among the four equations yields x^4 as a function of the x^i.

From the relation

$$dx^4 + p_i dx^i = 0,$$

identically verified from the solutions of equations (1.2) on the characteristic surface Σ_0, one infers that the partial derivatives of this function x^4 with respect to the x^i verify the relation

$$\frac{\partial x^4}{\partial x^i} = -p_i.$$

[3]Note that the integral equations (1.2) under consideration are nonlinear integral equations, the quantity under integration symbol being a polynomial of given functions of the unknown functions. It is easy to prove that these equations have a continuous, bounded, three times differentiable solution, verifying

$$\left|x^i - \overline{x}^i\right| \leq d \text{ and } \left|x^4\right| \leq \varepsilon$$

within the domain (Λ) below. Analogous proofs are performed in chapter III.

If we denote by $[\varphi]$ the value of a function φ of four coordinates x^α on Σ_0 and if we express $[\varphi]$ as a function of the three parameters x^i representatives of Σ_0, the partial derivatives of this functio with respect to the x^i fulfill therefore:

$$\frac{\partial[\varphi]}{\partial x^i} = \left[\frac{\partial \varphi}{\partial x^i}\right] - \left[\frac{\partial \varphi}{\partial x^4}\right] p_i. \tag{2.3}$$

3 Integral equations satisfied from the derivatives of the functions $x^i(\lambda)$ and $p_i(\lambda)$

We shall set

$$\frac{\partial x^i}{\partial p_j^0} = y_j^i,\ \frac{\partial^2 x^i}{\partial p_j^0 \partial p_h^0} = y_{jh}^i,\ \frac{\partial^3 x^i}{\partial p_j^0 \partial p_h^0 \partial p_k^0} = y_{jhk}^i,$$

$$\frac{\partial p_i}{\partial p_j^0} = z_j^i,\ \frac{\partial^2 p_i}{\partial p_j^0 \partial p_h^0} = z_{jh}^i,\ \frac{\partial^3 p_i}{\partial p_j^0 \partial p_h^0 \partial p_k^0} = z_{jhk}^i.$$

These functions satisfy the integral equations obtained by derivation under the summation sign with respect to the p_i^0 of equations (1.2) (the quantities obtained under the integration signs being continuous and bounded). Formula (2.3) shows that these equations can be written (the derivatives $\frac{\partial x^4}{\partial p_i^0}$ being useless)

$$y_j^i = \int_0^{\lambda_1} T_j^i d\lambda_1,\ T_j^i = \frac{\partial T^i}{\partial p_j^0} = \left\{ \frac{\partial}{\partial x^h}[A^{ih}]p_h + \frac{\partial}{\partial x^4}[A^{i4}] \right\} y_j^h + [A^{ih}] z_j^h,$$

$$z_j^i = \int_0^{\lambda_1} R_j^i d\lambda_1,\ R_j^i = \frac{\partial R_i}{\partial p_j^0} = \frac{\partial R_i}{\partial x_k} y_j^k + \frac{\partial R_i}{\partial p_k} z_j^k,$$

$$y_{jk}^i = \int_0^{\lambda_1} T_{jk}^i d\lambda_1,\ T_{jk}^i = \frac{\partial T_0^i}{\partial p_k^0} = \frac{\partial T_i}{\partial x^h} y_{jk}^h + \frac{\partial T^i}{\partial p_h} z_{jk}^h + \phi_{jk}^i,$$

$$z_{jk}^i = \int_0^{\lambda_1} R_{jk}^i d\lambda_1,\ R_{jk}^i = \frac{\partial R_j^i}{\partial p_k^0} = \frac{\partial R_i}{\partial x^h} y_{jk}^h + \frac{\partial R_i}{\partial p_h} z_{jk}^h + \psi_{jk}^i,$$

where ϕ_{jk}^i and ψ_{jk}^i are polynomials of the functions $p_i(\lambda), y_j^i(\lambda), z_j^i(\lambda)$, of the coefficients $A^{\lambda\mu}(x^\alpha)$ and of their partial derivatives with respect to the x^α up to the third order included. In these functions the x^α are replaced from the $x^\alpha(\lambda)$ given by the formulas (2.1).

We would find by analogous fashion

$$y_{jhk}^i = \int_0^{\lambda_1} T_{jhk}^i d\lambda_1,\ T_{jhk}^i = \frac{\partial T_{jh}^i}{\partial p_k^0} = \frac{\partial T^i}{\partial x^l} y_{jhk}^l + \frac{\partial T^i}{\partial p^l} z_{jhk}^l + \phi_{jhk}^i,$$

$$z_{jhk}^i = \int_0^{\lambda_1} R_{jhk}^i d\lambda_1,\ R_{jhk}^i = \frac{\partial R_{jh}^i}{\partial p_k^0} = \frac{\partial R^i}{\partial x^l} y_{jhk}^l + \frac{\partial R^i}{\partial p^l} z_{jhk}^l + \psi_{jhk}^i,$$

where ϕ_{jhk}^i and ψ_{jhk}^i are polynomials of the functions $p_i, y_j^i, z_j^i, y_{jh}^i, z_{jh}^i$ as well as of the coefficients $A^{\lambda\mu}$ and of their partial derivatives up to the fourth order included (functions of the functions x^α).

4 Relations satisfied by the unknown functions on the surface of the characteristic conoid

We will denote by $[\varphi]$ the value of a function φ of the four coordinates x^α on the surface of the characteristic conoid Σ_0. $[\varphi]$ can be expressed as a function of three variables of a parametric representation of Σ_0, in particular of the three coordinates x^i. In light of the equality (2.3) the partial derivatives of this function with respect to the x^i verify the relation

$$\left[\frac{\partial[\varphi]}{\partial x^i}\right] = \left[\frac{\partial \varphi}{\partial x^i}\right] - \left[\frac{\partial \varphi}{\partial x^4}\right] p_i.$$

One applies again this rule to the evaluation of the derivatives

$$\frac{\partial}{\partial x^i}\left[\frac{\partial \varphi}{\partial x^i}\right] \text{ and } \frac{\partial}{\partial x^j}\left[\frac{\partial \varphi}{\partial x^4}\right],$$

from which it follows easily

$$\left[\frac{\partial^2 \varphi}{\partial x^i \partial x^4}\right] = \frac{\partial}{\partial x^i}\left[\frac{\partial \varphi}{\partial x^4}\right] + \left[\frac{\partial^2 \varphi}{\partial {x^4}^2}\right] p_i,$$

$$\left[\frac{\partial^2 \varphi}{\partial x^i \partial x^j}\right] = \frac{\partial^2 [\varphi]}{\partial x^i \partial x^j} + \frac{\partial}{\partial x^i}\left[\frac{\partial \varphi}{\partial x^4}\right] p_j + \frac{\partial}{\partial x^j}\left[\frac{\partial \varphi}{\partial x^4}\right] p_i + \left[\frac{\partial \varphi}{\partial x^4}\right] \frac{\partial p_i}{\partial x^j}$$
$$+ \left[\frac{\partial^2 \varphi}{\partial {x^4}^2}\right] p_i p_j.$$

These identities make it possible to write the following relations satisfied by the unknown functions u_s on the characteristic conoid:

$$\begin{aligned} [E_r] &= [A^{ij}]\frac{\partial^2 [u_r]}{\partial x^i \partial x^j} + \left\{[A^{ij}]p_i p_j + 2[A^{i4}]p_i + [A^{44}]\right\} \frac{\partial^2 u_r}{\partial {x^4}^2} \\ &+ 2\left\{[A^{ij}]p_j + [A^{i4}]\right\} \frac{\partial}{\partial x^i}\left[\frac{\partial u_r}{\partial x^4}\right] + \left[\frac{\partial u_r}{\partial x^4}\right][A^{ij}]\frac{\partial p_i}{\partial x^j} + B_r^{s\mu}\left[\frac{\partial u_s}{\partial x^\mu}\right] \\ &+ [f_r] = 0. \end{aligned} \tag{4.1}$$

The coefficient of the term $\left[\frac{\partial^2 u_r}{\partial {x^4}^2}\right]$ is the value on the characteristic conoid of the first member of equation (1.1); it therefore vanishes. We might have expected on the other hand that the equations $[E_r] = 0$ would not contain second derivatives of the functions u_r but those obtained by derivation on the surface Σ_0, the assignment on a characteristic surface of the unknown functions $[u_r]$ and of their first derivatives $\left[\frac{\partial u_r}{\partial x^\alpha}\right]$ not being able to determine the set of second derivatives.

B. Auxiliary functions

5 Introduction of the auxiliary functions σ_s^r. Occurrence of a divergence

We form n^2 linear combinations $\sigma_s^r[E_r]$ of the equations (4.1) verified by the unknown functions within the domain V of Σ_0, the σ_s^r denoting n^2 auxiliary functions which possess at M_0 a aingularity.

We set

$$M(\varphi) = [A^{ij}]\frac{\partial^2 \varphi}{\partial x^i \partial x^j},$$

φ denoting a function whatsoever of the three variables x^i, and we write

$$\begin{aligned} \sigma_s^r E_r &= \Big\{M([u_r]) + 2([A^{ij}]p_j + [A^{i4}])\frac{\partial}{\partial x^i}\left[\frac{\partial u_r}{\partial x^4}\right] \\ &+ \left[\frac{\partial u_r}{\partial x^4}\right][A^{ij}]\frac{\partial p_i}{\partial x^j} + [B_r^{t\mu}]\left[\frac{\partial u_t}{\partial x^\mu}\right] + [f_r]\Big\}\sigma_s^r = 0. \end{aligned} \tag{5.1}$$

We will transform these equations in such a way that a divergence occurs therein, whose volume integral will get transformed into a surface integral, while the remaining terms will contain only $[u_r]$ and $\left[\frac{\partial u_r}{\partial x^4}\right]$. We will use for that purpose the following identity, verified by two functions whatsoever φ and ψ of the three variables x^i:

$$\psi M(\varphi) = \frac{\partial}{\partial x^i}\left([A^{ij}]\psi \frac{\partial \varphi}{\partial x^j}\right) - \frac{\partial \varphi}{\partial x^j}\frac{\partial}{\partial x^i}([A^{ij}]\psi)$$

or

$$\psi M(\varphi) = \frac{\partial}{\partial x^i}\left([A^{ij}]\psi \frac{\partial \varphi}{\partial x^j} - \varphi \frac{\partial}{\partial x^j}([A^{ij}]\varphi)\right) + \varphi \overline{M}(\psi),$$

where $\overline{M}$ is the adjoint operator of M, i.e.

$$\overline{M}(\psi) = \frac{\partial^2([A^{ij}]\psi)}{\partial x^i \partial x^j},$$

and the identity (2.3), previously written, which yields here

$$\left[\frac{\partial u_r}{\partial x^i}\right] = \frac{\partial [u_r]}{\partial x^i} + p_i \left[\frac{\partial u_r}{\partial x^4}\right].$$

We see without difficulty that the expressions $\sigma_s^r[E_r]$ take the form

$$\sigma_s^r[E_r] = \frac{\partial}{\partial x^i}E_s^i + [u_r]L_s^r + \sigma_s^r f_r - \left[\frac{\partial u_r}{\partial x^4}\right] D_s^r,$$

where one has defined

$$\begin{aligned}
E_s^i &= [A^{ij}]\sigma_s^r \frac{\partial [u_r]}{\partial x^j} - [u_r]\frac{\partial}{\partial x^j}([A^{ij}]\sigma_s^r) + 2\sigma_s^r \left\{[A^{ij}]p_j + [A^{i4}]\right\}\left[\frac{\partial u_r}{\partial x^4}\right] \\
&\quad + [B_{ri}^t][u_t]\sigma_s^r, \\
L_s^r &= \overline{M}(\sigma_s^r) - \frac{\partial}{\partial x^i}\left([B_t^{ri}]\sigma_s^t\right), \qquad (5.2) \\
D_s^r &= \sigma_s^r \left\{2\frac{\partial}{\partial x^i}([A^{ij}]p_j + [A^{i4}]) - [A^{ij}]\frac{\partial p_j}{\partial x^i}\right\} + 2([A^{ij}]p_j + [A^{i4}])\frac{\partial \sigma_s^r}{\partial x^i} \\
&\quad - ([B_t^{r4}] + [B_t^{ri}]p_i)\sigma_s^t.
\end{aligned}$$

We will choose the auxiliary functions σ_s^r in such a way that, in every equation, the coefficient of $\left[\frac{\partial u_r}{\partial x^4}\right]$ vanishes. These functions will therefore have to fulfill n^2 partial differential equations of first order

$$D_s^r = 0. \qquad (5.3)$$

We will see that these equations possess a solution having at M_0 the desired singularity. If the auxiliary functions σ_s^r verify these n^2 relations, the equations, verified by the unknown functions u_r on the characteristic conoid Σ_0, take the simple form

$$[u_r]L_s^r + \sigma_s^r[f_r] + \frac{\partial}{\partial x^i}E_s^i = 0. \qquad (5.4)$$

6 Integration of the obtained equations

We will integrate the equations so obtained with respect to the three variables x^i on a portion V_η of hypersurface of the characteristic conoid Σ_0, limited by the hypersurfaces $x^4 = 0$ and $x^4 = x_0^4 - \eta$. This domain V_η is defined to be simply connected and internal to the domain V if the coordinate x_0^4 is sufficiently small. As a matter of fact:

$$\left|x_0^4\right| < \varepsilon_0 \text{ implies within } V_\eta \ \left|x^4 - x_0^4\right| < \varepsilon_0.$$

The formula (2.2) shows in such a case that, for a suitable choice of ε_0, we will have

$$\lambda_1 \leq \varepsilon_1.$$

Since the boundary of V_η consists of two-dimensional domains S_0 and S_η cut over Σ_0 from the hypersurfaces $x^4 = 0, x^4 = x_0^4 - \eta$ we will have, upon integrating the equations (5.4) within V_η, the following fundamental relations:

$$\begin{aligned} & \int\int_{V_\eta}\int \left\{[u_r]L_s^r + \sigma_s^r[f_r]\right\} dV + \int\int_{S_\eta} E_s^i \cos(n, x^i) dS \\ - & \int\int_{S_0} E_s^i \cos(n, x^i) dS = 0, \end{aligned} \tag{6.1}$$

where dV, dS and $\cos(n, x^i)$ denote, in the space of three variables x^i, the volume element, the area element of a surface $x^4 = C^{te}$ and the directional cosines of the outward-pointing normal to one of such surfaces, respectively.

The limit of these equations, when η tends to zero, will provide us with Kirchhoff formulas that we will build in the last part of this chapter.

7 Determination of the auxiliary functions σ_s^r

We will look for a solution of equations (5.3) in the form

$$\sigma_s^r = \sigma \omega_s^r,$$

where σ is infinite at the point M_0 and the ω_s^r are bounded.

The equations (5.3) read as

$$\sigma_s^r \left\{ \frac{\partial}{\partial x^i}([A^{ij}]p_j + [A^{i4}]) + p_j \frac{\partial}{\partial x^i}[A^{ij}] + \frac{\partial}{\partial x^i}[A^{i4}] \right\}$$

$$-([B_t^{r4}] + [B_t^{ri}]p_i)\sigma_s^r + 2([A^{ij}]p_j + [A^{i4}]) \frac{\partial \sigma_s^r}{\partial x^i} = 0.$$

The coefficients $A^{\lambda\mu}, B_s^{t\lambda}$, the first derivatives of the $A^{\lambda\mu}$ and the functions p_i are bounded within the domain V, the coefficients of the linear first-order partial differential equations are therefore a sum of bounded terms, perhaps with exception of the terms

$$\frac{\partial}{\partial x^i} \left\{[A^{ij}]p_j + [A^{i4}]\right\}.$$

We will therefore choose the ω_s^r, that we want to be bounded, as satisfying the equation

$$\omega_s^r p_j \frac{\partial}{\partial x^i}[A^{ij}] + \frac{\partial}{\partial x^i}[A^{i4}] - \omega_s^t \left\{[B_t^{r4}] + [B_t^{ri}]p_i\right\} + 2\left\{[A^{ij}]p_j + [A^{i4}]\right\} \frac{\partial \omega_s^r}{\partial x^i} = 0, \tag{7.1}$$

fulfilling in turn

$$\sigma \frac{\partial}{\partial x^i}\Big([A^{ij}]p_j + [A^{i4}]\Big) + 2\Big([A^{ij}]p_j + [A^{i4}]\Big) \frac{\partial \sigma}{\partial x^i} = 0. \tag{7.2}$$

8 Determination of the ω_s^r

We see easily that the equations (7.1) can be written in form of integral equations analogous to the equations (1.2) obtained in the search for the conoid Σ_0. We have indeed, on Σ_0:

$$[A^{ij}]p_j + [A^{i4}] = T^i = \frac{\partial x^i}{\partial \lambda_1},$$

from which, for an arbitrary function φ defined on Σ_0,

$$T^i \frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial \lambda_1}.$$

Let us impose upon the ω_s^r the limiting conditions

$$\omega_s^r = \delta_s^r \text{ for } \lambda_1 = 0.$$

These quantities satisfy therefore the integral equations

$$\omega_s^r = \int_0^{\lambda_1} (Q_t^r \omega_s^t + q\omega_s^r) d\lambda_1 + \delta_s^r \tag{8.1}$$

with

$$Q_t^r = \frac{1}{2}([B_t^{r4}] + [B_t^{ri}]p_i) \text{ and } Q = -\frac{1}{2}\left(p_j \frac{\partial}{\partial x^i}[A^{ij}] + \frac{\partial}{\partial x^i}[A^{i4}]\right),$$

the assumptions made upon the coefficients $A^{\lambda\mu}$ and $B_s^{r\lambda}$ and the results obtained on the functions x^i, p_i enabling moreover to prove that, for a convenient choice of ϵ_1, these equations have a unique, continuous, bounded solution which has partial derivatives of the first two orders with respect to the p_i^0, continuous and bounded within the domain Λ. We will denote these derivatives by ω_{si}^r and ω_{sij}^r.

9 Determination of σ

Let us consider the equation (7.2) verified by σ. We know that

$$([A^{ij}p_j + [A^{i4}])\frac{\partial \sigma}{\partial x^i} = \frac{\partial \sigma}{\partial \lambda_1},$$

and we are going to evaluate the coefficient of σ,

$$\frac{\partial}{\partial x^i}([A^{ij}]p_j + [A^{i4}]),$$

by relating it very simply to the determinant

$$\triangle = \frac{D(x^1, x^2, x^3)}{D(\lambda_1, \lambda_2, \lambda_3)}.$$

This determinant $\triangle$, Jacobian of the change of variables $x^i = x^i(\lambda_j)$ on the conoid Σ_0, has for elements

$$\frac{\partial x^i}{\partial \lambda_1} = T^i,\ \frac{\partial x^i}{\partial \lambda_2} = y_j^i \frac{\partial p_j^0}{\partial \lambda_2},\ \frac{\partial x^i}{\partial \lambda_3} = y_j^i \frac{\partial p_j^0}{\partial \lambda_3}.$$

Let us denote by $\triangle_j^i$ the minor relative to the element $\frac{\partial x^i}{\partial \lambda_j}$ of the determinant $\triangle$.

A function whatsoever φ, defined on Σ_0, verifies the identities

$$\frac{\partial \varphi}{\partial x^i} = \frac{\triangle_i^j}{\triangle}\frac{\partial \varphi}{\partial \lambda_j}.$$

Let us apply this formula to the function $\frac{\partial x^i}{\partial \lambda_1} = T^i$:

$$\frac{\partial}{\partial x^i} T^i = \frac{\triangle_j^i}{\triangle} \frac{\partial}{\partial \lambda_j} T^i = \frac{\triangle_i^j}{\triangle} \frac{\partial}{\partial \lambda_1} \left(\frac{\partial x^i}{\partial \lambda_j} \right),$$

$\triangle_i^j$ being the minor relative to the element $\frac{\partial x^i}{\partial \lambda_j}$ of the determinant $\triangle$ we have

$$\frac{\partial}{\partial x^i} T^i = \frac{1}{\triangle} \frac{\partial \triangle}{\partial \lambda_1}.$$

Thus, the function σ verifies the relation

$$\sigma \frac{\partial \triangle}{\partial \lambda_1} + 2 \triangle \frac{\partial \sigma}{\partial \lambda_1} = 0$$

which is integrated in immediate way. The general solution is

$$\sigma = \frac{f(\lambda_2, \lambda_3)}{\mid \triangle \mid^{\frac{1}{2}}},$$

where f denotes an arbitrary function.

For λ_1 the determinant $\triangle$ vanishes, because the y_j^i are vanishing; the function σ is therefore infinite.

The coefficients $A^{\lambda\mu}$ and their first and second partial derivatives with respect to the x^α being continuous and bounded within the domain V of Σ_0, as well as the functions x^i, y_j^i, z_j^i, we have

$$\lim_{\lambda_1 \to 0} \frac{y_j^i}{\lambda_1} = [A^{ij}]_{\lambda_1 = 0} = -\delta_i^j. \tag{9.1}$$

By dividing the second and third line of $\triangle$ by λ_1 we obtain a determinant equal to $\frac{\triangle}{\lambda_1^2}$; we deduce from the formulas (9.1) and (9.2)

$$\lim_{\lambda_1 \to 0} \frac{\triangle}{\lambda_1^2} = \det \begin{pmatrix} -\sin\lambda_2 \cos\lambda_3 & -\sin\lambda_2 \sin\lambda_3 & -\cos\lambda_2 \\ -\cos\lambda_2 \cos\lambda_3 & -\cos\lambda_2 \sin\lambda_3 & \sin\lambda_2 \\ +\sin\lambda_2 \sin\lambda_3 & -\sin\lambda_2 \cos\lambda_3 & 0 \end{pmatrix} = -\sin\lambda_2.$$

As a matter of fact:

$$\lim_{\lambda_1 \to 0} T^i = -\delta_i^j p_j^0 = -p_i^0$$

$$\lim_{\lambda_1 \to 0} \frac{1}{\lambda_1} \frac{\partial x^i}{\partial \lambda_u} = \lim_{\lambda_1 \to 0} \frac{y_j^i}{\lambda_1} \frac{\partial p_j^0}{\partial \lambda_u} = -\delta_j^i \frac{\partial p_j^0}{\partial \lambda_u}.$$

We will take for auxiliary function σ the function

$$\sigma = \left| \frac{\sin\lambda_2}{\triangle} \right|^{\frac{1}{2}}.$$

We will then have $\lim_{\lambda_1 \to 0} \sigma \lambda_1 = 1$.

10 Derivatives of the functions σ_s^r

The equations (6.1) contain, on the one hand the values on Σ_0 of the unknown functions u_r, of their partial derivatives as well as the functions p_i, y and z, on the other hand the functions σ_s^r and their first and second partial derivatives.

Let us study therefore the partial derivatives of the first two orders of the functions σ and ω_s^r.

Derivatives of σ:

$$\sigma = \left|\frac{\sin\lambda_2}{\triangle}\right|^{\frac{1}{2}}$$

is a function of the trigonometric lines of λ_u $(u = 2, 3)$, of the functions x^α (through the intermediate effect of the $A^{\lambda\mu}$) and of the functions p_i, y^i_j. The first and second partial derivatives of σ with respect to the x^i will be therefore expressed with the help of the functions listed and of their first and second partial derivatives.

(1^o) First derivatives: We have seen that the partial derivatives with respect to the x^i of a function whatsoever φ, defined on Σ_0, satisfy the identity

$$\frac{\partial\varphi}{\partial x^i} = \frac{\triangle^j_i}{\triangle}\frac{\partial\varphi}{\partial\lambda_j}, \tag{10.1}$$

where $\frac{\triangle^j_i}{\triangle}$ is a given function of $\cos\lambda_u, \sin\lambda_u, x^\alpha, p_i, y^j_i$, the partial derivatives with respect to λ_1 of the functions x^i, p_i, y^j_i,[4] are the quantities T^i, R_i, T^i_j which are expressed through these functions themselves and through z^j_i, the partial derivatives with respect to λ_u of these functions x^i, p_i, y^j_i being expressible by means of their derivatives with respect to the overabundant parameters p^0_h, denoted here by y^i_h, z^i_h, y^j_{ih}, and by means of $\cos\lambda_u, \sin\lambda_u$.

The function σ admits therefore within V, under the assumptions made, of first partial derivatives with respect to the x^i which are expressible by means of the functions x^α (with the intermediate help of the $[A^{\lambda\mu}]$ and of the $\left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right]$ and of the functions $p_i, y^i_j, z^i_j, y^i_{jh}$ and of $\cos\lambda_u, \sin\lambda_u$).

(2^o) Second derivatives: A new application of the formula (10.1) shows, in analogous fashion, that σ admits within V of second partial derivatives, which are expressible by means of the functions x^α (with the intermediate action of the $A^{\lambda\mu}$ and their first and second partial derivatives) and of the functions $p_i, y^i_j, z^i_j, y^j_{ih}, z^j_{ih}, y^j_{ihk}$ and of $\cos\lambda_u, \sin\lambda_u$.

Derivatives of the ω^r_s: The identity (10.1) makes it possible moreover to show that the functions ω^r_s, solutions of the equations (7.1), admit within V of first and second partial derivatives with respect to the variables x^i if these functions admit, within V, of first and second partial derivatives with respect to the variables λ_u; it suffices for that purpose that they admit of first and second partial derivatives with respect to the overabundant variables p^0_i.

We shall set

$$\frac{\partial\omega^r_s}{\partial p^0_i} = \omega^r_{si}, \ \frac{\partial^2\omega^r_s}{\partial p^0_i\partial p^0_j} = \omega^r_{sij}.$$

If these functions are continuous and bounded within V they satisfy, under the assumptions made, the integral equations obtained by derivation under the summation symbol of the equations (8.1) with respect to the p^0_i. Let respectively

$$1^o) \quad \omega^r_{si} = \int_0^{\lambda_1}(Q^r_t\omega^t_{si} + Q\omega^r_{si} + \Omega^r_{si})d\lambda_1,$$

where

$$\Omega^r_{si} = \frac{\partial Q^r_t}{\partial p^0_i}\omega^t_s + \frac{\partial Q}{\partial p^0_i}\omega^r_s$$

is a polynomial of the functions $\omega^r_s, p_i, y^i_j, z^i_j$ as well as of the values on Σ_0 of the coefficients $A^{\lambda\mu}, B^{r\lambda}_s$ of the equations (E) and of their partial derivatives with respect to the x^α up to the orders two and one, respectively (quantities that are themselves functions of the functions $x^\alpha(\lambda_j)$).

$$2^o) \quad \omega^r_{sij} = \int_0^{\lambda_1}(Q^r_t\omega^t_{sij} + Q\omega^r_{sij} + \Omega^r_{sij})d\lambda_1,$$

[4] The partial derivatives of the function x^4 with respect to the variables x^i are known directly because $\frac{\partial x^4}{\partial x^i} = -p_i$.

where

$$\Omega^r_{sij} = \frac{\partial Q^r_t}{\partial p^0_j}\omega^t_{si} + \frac{\partial Q}{\partial p^0_j}\omega^r_{si} + \frac{\partial \Omega^r_{si}}{\partial p^0_j},$$

is a polynomial of the functions $\omega^r_s, \omega^r_{si}, p_i, y^j_i, z^j_i, y^j_{ih}, z^j_{ih}$ as well as of the values on Σ_0 of the coefficients $A^{\lambda\mu}, B^{r\lambda}_s$ and of their partial derivatives with respect to the x^α up to the orders three and two, respectively.

The first and second partial derivatives of the ω^r_s with respect to the variables x^i are expressed by means of the functions x^α (with the help of the coefficients $A^{\lambda\mu}$ and of their first partial derivatives), $p_i, y^i_j, z^i_j, y^i_{jh}, z^i_{jh}, \omega^r_s, \omega^r_{si}$ and ω^r_{sij}.

Summary. We have shown that the auxiliary functions σ^r_s exist and admit within V of first and second partial derivatives with respect to the variables x^i under the following assumptions:

(1^o) The coefficients $A^{\lambda\mu}$ and $B^{r\lambda}_s$ have partial derivatives continuous and bounded up to the orders four and two, respectively, within the domain D containing V.

(2^o) The integral equations for the unknown functions x^α, p_i and ω^r_s have a unique, continuous, bounded solution and admitting within V partial derivatives with respect to the p^0_i, continuous and bounded up to the second order. This result can be proved by assuming that the partial derivatives of order four and two, respectively, of the functions $A^{\lambda\mu}$ and $B^{r\lambda}_s$ verify some Lipschitz conditions.

The functions σ^r_s and their first and second partial derivatives with respect to the x^i are then expressed only through some functions X and Ω, X denoting any whatsoever of the functions $x^\alpha, p_i, y^j_i, z^j_i, y^j_{ih}, z^j_{ih}, y^j_{ihk}, z^j_{ihk}$ and ω any whatsoever among the functions $\omega^r_s, \omega^r_{si}, \omega^r_{sij}$.

The functions X and Ω satisfy integral equations of the form

$$X = \int_0^{\lambda_1} E(X) d\lambda_1 + X_0,$$

$$\Omega = \int_0^{\lambda_1} F(X, \Omega) d\lambda_1 + \Omega_0,$$

where X_0 and Ω_0 denote the given values of the functions X and Ω for $\lambda_1 = 0$.

$E(X)$ is a polynomial of the functions X and of the values on Σ_0 of the coefficients $A^{\lambda\mu}$ and of their partial derivatives up to the fourth order (functions of the functions x^α).

$F(X, \Omega)$ is a polynomial of the functions X and Ω, and of the values on Σ_0 of the coefficients $A^{\lambda\mu}, B^{r\lambda}_s$ and of their partial derivatives up to the orders three and two, respectively.

11 Studies of the behaviour in the neighbourhood of the vertex of the characteristic conoid

We are going to study the quantities occurring in the integrals of the fundamental relations (6.1), and for this purpose we will look in a more precise way for the expression of the partial derivatives of the functions σ and ω^r_s with respect to the variables x^i by means of the functions X and Ω. The behaviour of these functions in the neighbourhood of $\lambda_1 = 0$ (vertex of the characteristic conoid Σ_0) will make it possible for us to look for the limit of equations (6.1) for $\eta = 0$: the function $x^4(\lambda_1, \lambda_2, \lambda_3)$ being, within the domain Λ, a continuous function of the three variables λ_i, $\eta = x^4 - x^4_0$ tends actually to zero with λ_1. We will provide the details of the calculations, that we will need in the following, when we will try to solve the system of integral equations obtained.

We will use essentially in the studies of the behaviour in the neighbourhood of $\lambda_1 = 0$, the following fact which results from the assumptions made and from the equations verified by the functions $y^i_j, y^i_{jh}, y^i_{jhk}, \omega^r_{si}$ and ω^r_{sij}.

The functions $\frac{y^i_j}{\lambda_1}, \frac{y^i_{jh}}{\lambda_1}, \frac{y^i_{jhk}}{\lambda_1}$, and $\frac{\omega^r_{si}}{\lambda_1}, \frac{\omega^r_{sij}}{\lambda_1}$ are continuous and bounded functions of $\lambda_1, \lambda_2, \lambda_3$ within the domain V. We will denote any whatsoever of these functions by $\tilde{X}$ and $\tilde{\Omega}$.

12 Behaviour in the neighbourhood of $\lambda_1 = 0$ of the determinant $\triangle$ and of its minors

(1^o) We have already shown (Sec. 9) that the quantity $\frac{\triangle}{\lambda_1^2}$ is a polynomial of the functions X (here p_i only), $\tilde{X}$ (here $\frac{y^i_j}{\lambda_1}$ only), of the coefficients $A^{\lambda\mu}$ and of the $\sin\lambda_u, \cos\lambda_u$ ($u = 2, 3$). It is therefore a continuous bounded function of $\lambda_1, \lambda_2, \lambda_3$ within V. We have seen that the value of this function for $\lambda_1 = 0$ is

$$\lim_{\lambda_1 \to 0} \frac{\triangle}{\lambda_1^2} = -\sin\lambda_2.$$

In the neighbourhood of $\lambda_1 = 0$ the function $\frac{\triangle}{\lambda_1^2}$, which will occur in the denominator of the quanties studied in the following, is $\neq 0$, but for $\lambda_2 = 0$ or $\lambda_2 = \pi$. To remove this difficulty we will show that the polynomial $\triangle$ is divisible by $\sin\lambda_2$ and we will make sure that the function $D = \frac{\triangle}{\lambda_1^2 \sin\lambda_2}$ appears in the denominators we consider.

Let us therefore consider on the conoid Σ_0 the following change of variables:

$$\mu_i = \lambda_1 p_i^0. \tag{12.1}$$

We set

$$d = \frac{D(\mu_1, \mu_2, \mu_3)}{D(\lambda_1, \lambda_2, \lambda_3)} = \det \begin{pmatrix} p_1^0 & p_2^0 & p_3^0 \\ \lambda_1 \frac{\partial p_1^0}{\partial \lambda_2} & \lambda_1 \frac{\partial p_2^0}{\partial \lambda_2} & \lambda_1 \frac{\partial p_3^0}{\partial \lambda_2} \\ \lambda_1 \frac{\partial p_1^0}{\partial \lambda_3} & \lambda_1 \frac{\partial p_2^0}{\partial \lambda_3} & \lambda_1 \frac{\partial p_3^0}{\partial \lambda_3} \end{pmatrix} = \lambda_1^2 \sin\lambda_2$$

and

$$\triangle = \frac{D(x^1, x^2, x^3)}{D(\lambda_1, \lambda_2, \lambda_3)}.$$

Since

$$\frac{D(x^1, x^2, x^3)}{D(\lambda_1, \lambda_2, \lambda_3)} = \frac{D(x^1, x^2, x^3)}{D(\mu_1, \mu_2, \mu_3)} \frac{D(\mu_1, \mu_2, \mu_3)}{D(\lambda_1, \lambda_2, \lambda_3)}$$

we have

$$\triangle = D\lambda_1^2 \sin\lambda_2, \tag{12.2}$$

where the determinant D has elements

$$\frac{\partial x^i}{\partial \mu_j} = \frac{\partial x^i}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial \mu_j} + \frac{\partial x^i}{\partial p_h^0} \frac{\partial p_h^0}{\partial \lambda_u} \frac{\partial \lambda_u}{\partial \mu_j}.$$

It results directly from the equalities (12.1) and from the identity $\sum_i \mu_i^2 = \lambda_1^2$ that

$$\frac{\partial \lambda_1}{\partial \mu_j} = p_j^0 \text{ and } \frac{\partial p_h^0}{\partial \lambda_u} = \frac{1}{\lambda_1} \frac{\partial \mu_h}{\partial \lambda_u}.$$

On the other hand we have

$$\frac{\partial \lambda_1}{\partial \mu_j} \frac{\partial \mu_h}{\partial \lambda_1} + \frac{\partial \lambda_u}{\partial \mu_j} \frac{\partial \mu_h}{\partial \lambda_u} = \delta_j^h.$$

The elements of D are therefore

$$\frac{\partial x^i}{\partial \mu_j} = T^i p_j^0 + \frac{y_h^i}{\lambda_1} (\delta_j^h - p_j^0 p_h^0).$$

The polynomial $\frac{\triangle}{\lambda_1^2}$ is therefore divisible by $\sin\lambda_2$, the quotient D being a polynomial of the same functions $X, \tilde{X}$ as $\frac{\triangle}{\lambda_1^2}$ is of $\sin\lambda_u, \cos\lambda_u$ (or, more precisely, of the three p_i^0).

D is a continuous bounded function of $\lambda_1, \lambda_2, \lambda_3$ within V whose value for $\lambda_1 = 0$ is $\lim_{\lambda_1 \to 0} D = -1$. As a matter of fact:

$$\lim_{\lambda_1 \to 0} \frac{\partial x^i}{\partial \mu_j} = -p_i^0 p_j^0 - \delta_j^i + p_i^0 p_j^0 = -\delta_j^i.$$

Remark. $\frac{\triangle}{\lambda_1^2}$ being a homogeneous polynomial of the second degree of the functions $\frac{y_i^j}{\lambda_1}$, the same is true of the polynomial D, and the quantity $\lambda_1^2 D$ is a polynomial of the functions X (p_i and y_i^j), of the coefficients $A^{\lambda\mu}$ and of the three p_i^0, homogeneous of the second degree with respect to the y_i^j. One can easily verify these results by evaluating the product $D^+ d^+$ where

$$d^+ = \det \begin{pmatrix} p_1^0 & p_2^0 & p_3^0 \\ \frac{\partial p_1^0}{\partial \lambda_2} & \frac{\partial p_2^0}{\partial \lambda_2} & \frac{\partial p_3^0}{\partial \lambda_2} \\ \frac{\partial p_1^0}{\partial \lambda_3} & \frac{\partial p_2^0}{\partial \lambda_3} & \frac{\partial p_3^0}{\partial \lambda_3} \end{pmatrix} = \frac{d}{\lambda_1^2}$$

and where D^+ is the determinant whose elements are

$$T^i p_j^0 - y_h^i (\delta_j^h - p_j^0 p_h^0).$$

One finds

$$D^+ d^+ = \triangle,$$

the quantity $\lambda_1^2 D = D^+$ possesses therefore the stated properties.

The polynomial D is, in absolute value, bigger than a number assigned in a domain W: D is actually a continuous and bounded function of λ_1 in the domain Λ (where λ_2 and λ_3 vary over a compact) which takes the value -1 for $\lambda_1 = 0$. There exists therefore a number ε_2 such that, in the domain Λ_2, neighbourhood of $\lambda_1 = 0$ of the domain Λ, defined by

$$|\lambda_1| \leq \varepsilon_2, \; 0 \leq \lambda_2 \leq \pi, \; 0 \leq \lambda_3 \leq 2\pi,$$

one has for example

$$|D + 1| \leq \frac{1}{2} \text{ therefore } |D| \geq \frac{1}{2}.$$

We will denote by W the domain of Σ_0 corresponding to the domain Λ_2.

(2^o) *Behaviour of the minors of* $\triangle$.

(a) Minors relative to the elements of the first line of $\triangle$: $\triangle_i^1$ is, as $\triangle$ itself, a homogeneous polynomial of second degree with respect to the functions y_i^j, and $\frac{\triangle_i^1}{\lambda_1^2}$ is a polynomial of the functions $X(p_i), \tilde{X}\left(\frac{y_i^j}{\lambda_1}\right)$, of the coefficients $[A^{\lambda\mu}]$ and of $\sin \lambda_u, \cos \lambda_u$; it is therefore a continuous and bounded function of $\lambda_1, \lambda_2, \lambda_3$ in V.

In order to study the quantity $\frac{\triangle_i^1}{\triangle} = \frac{\partial \lambda_1}{\partial x^i}$, which will occur in the following, we shall put it in the form of a rational fraction with denominator D ($\neq 0$ in W).

We have

$$\frac{\triangle_i^1}{\triangle} = \frac{\partial \lambda_1}{\partial x^i} = \frac{\partial \lambda_1}{\partial \mu_j} \frac{\partial \mu_j}{\partial x^i} = p_j^0 \frac{D_i^j}{D} \tag{12.3}$$

(one has denoted by D_i^j the minor relative to the element $\frac{\partial x^i}{\partial \mu_j}$ of the determinant D).

The quantity $\frac{\triangle_i^1}{\triangle}$ is therefore a continuous and bounded function of the three variables $\lambda_1, \lambda_2, \lambda_3$ in W. Let us compute the value of this function for $\lambda_1 = 0$, one finds

$$\lim_{\lambda_1 \to 0} \frac{\triangle_i^1}{\triangle} = -p_i^0,$$

a result that one might have expected. Indeed:

$$\lim_{\lambda_1 \to 0} \frac{\partial \lambda_1}{\partial x^i} = \lim_{\lambda_1 \to 0} \frac{\partial x^4}{\partial x^i},$$

or one has constantly, over Σ_0, $\frac{\partial x^4}{\partial x^i} = -p_i$.

Remark. One deduces from the formulas (12.2) and (12.3) that

$$\triangle_i^1 = \lambda_1^2 \sin\lambda_2 p_j^0 D_i^j.$$

One then sees that the quantity $\lambda_1^2 p_j^0 D_i^j$ is a polynomial of the functions p_i, y_i^j, of the coefficients $[A^{\lambda\mu}]$ and of the three p_h^0, homogeneous of second degree with respect to the y_i^j.

(b) Minors relative to the second and third line of $\triangle$: $\triangle_i^u$ is a polynomial of the functions $X(p_i, y_i^j), [A^{\lambda\mu}]$ and of $\sin\lambda_u, \cos\lambda_u$, homogeneous of first degree with respect to the functions y_i^j. $\frac{\triangle_i^u}{\lambda_1}$ is a continuous and bounded function of $\lambda_1, \lambda_2, \lambda_3$ in V.

Let us study the quantity $\frac{\partial p_h^0}{\partial\lambda_u}\frac{\triangle_i^u}{\triangle}$. One has

$$\frac{\partial p_h^0}{\partial\lambda_u}\frac{\triangle_i^u}{\triangle} = \frac{\partial p_h^0}{\partial\lambda_u}\frac{\partial\lambda_u}{\partial x^i} = \frac{1}{\lambda_1}\frac{\partial\mu_h}{\partial\lambda_u}\frac{\partial\lambda_u}{\partial\mu_j}\frac{\partial\mu_j}{\partial x^i} = \frac{1}{\lambda_1}(\delta_j^h - p_j^0 p_h^0)\frac{D_i^j}{D}.$$

We see that the quantity $\lambda_1\frac{\partial p_h^0}{\partial\lambda_u}\frac{\triangle_i^u}{\triangle}$ is a rational fraction with nonvanishing denominator (in the domain W) of the functions $X(p_i), \tilde{X}\left(\frac{y_i^j}{\lambda_1}\right), [A^{\lambda\mu}]$ and of the three p_i^0. It is therefore a continuous and bounded function of $\lambda_1, \lambda_2, \lambda_3$ in the domain W; the value of this function for $\lambda_1 = 0$ is computed as follows. One has on one hand

$$\frac{\partial x^h}{\partial\lambda_u} = \frac{\partial x^h}{\partial p_j^0}\frac{\partial p_j^0}{\partial\lambda_u} = y_j^h\frac{\partial p_j^0}{\partial\lambda_u},$$

$$\text{from which } \lim_{\lambda_1\to 0}\frac{1}{\lambda_1}\frac{\partial x^h}{\partial\lambda_u} = -\delta_j^h\frac{\partial p_j^0}{\partial\lambda_u} = -\frac{\partial p_h^0}{\partial\lambda_u}.$$

One knows on the other hand that

$$\frac{\triangle_i^u}{\triangle} = \frac{\partial\lambda_u}{\partial x^i},$$

$$\text{from which } \lim_{\lambda_1\to 0}\lambda_1\frac{\partial p_h^0}{\partial\lambda_u}\frac{\triangle_i^u}{\triangle} = -\lim_{\lambda_1\to 0}\frac{\partial x^h}{\partial\lambda_u}\frac{\partial\lambda_u}{\partial x^i} = -\delta_i^h + \lim_{\lambda_1\to 0}\frac{\partial x^h}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial x^i},$$

from which eventually

$$\lim_{\lambda_1\to 0}\lambda_1\frac{\partial p_h^0}{\partial\lambda_u}\frac{\triangle_i^u}{\triangle} = -\delta_i^h + p_i^0 p_h^0.$$

Remark. By a reasoning analogous to the one of previous remarks, one sees that the quantity $\lambda_1(\delta_j^h - p_j^0 p_h^0)D_i^j$ is a polynomial homogeneous of first degree with respect to the y_i^j, of the functions $X(p_i, y_i^j), [A^{\lambda\mu}], p_i^0$.

13 First derivatives

The first partial derivatives of an arbitrary function φ satisfy, in light of the identity (10.1) and of the results of the previous section, the relation

$$\frac{\partial\varphi}{\partial x^i} = \frac{\partial\varphi}{\partial\lambda_1}\frac{p_i^0 D_i^j}{D} + \frac{1}{\lambda_1}\frac{\partial\varphi}{\partial p_h^0}(\delta_j^h - p_j^0 p_h^0)\frac{D_i^j}{D}.$$

Let us apply this formula to the functions p_h^0 and X:

$$\frac{\partial p_h^0}{\partial x^i} = \frac{1}{\lambda_1}(\delta_j^h - p_j^0 p_h^0)\frac{D_i^j}{D},$$

$$\frac{\partial p_h}{\partial x^i} = R_h \frac{p_j^0 D_i^j}{D} + \delta_k^h \frac{1}{\lambda_1}(\delta_j^h - p_j^0 p_h^0)\frac{D_i^j}{D}, \tag{13.1}$$
$$\frac{\partial y_h^k}{\partial x^i} = T_h^k \frac{p_j^0 D_i^j}{E} + \frac{1}{\lambda_1} y_{hl}^k (\delta_j^h - p_j^0 p_h^0)\frac{D_i^j}{D},$$
$$\frac{\partial z_h^k}{\partial x^i} = R_h^k p_j^0 \frac{D_i^j}{D} + \frac{1}{\lambda_1} z_{hl}^k (\delta_j^l - p_j^0 p_l^0)\frac{D_i^j}{D}.$$

These equations and the analogous equations verified by

$$\frac{\partial y_{hl}^k}{\partial x^i}, \frac{\partial z_{hl}^k}{\partial x^i}, \frac{\partial \omega_s^r}{\partial x^i}, \frac{\partial \omega_{si}^r}{\partial x^i}$$

show that the quantities

$$\lambda_1 \frac{\partial p_h^0}{\partial x^i},\ \lambda_1 \frac{\partial p_h}{\partial x^i},\ \lambda_1 \frac{\partial z_h^k}{\partial x^i},\ \lambda_1 \frac{\partial z_{hl}^k}{\partial x^i},\ \text{and}\ \frac{\partial y_h^k}{\partial x^i},\ \frac{\partial y_{hl}^k}{\partial x^i},\ \frac{\partial \omega_s^r}{\partial x^i},\ \frac{\partial \omega_{si}^r}{\partial x^i}$$

are rational fractions with denominator D of the functions

$$X, \tilde{X}, \Omega, \tilde{\Omega}, [A^{\lambda\mu}], \left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right], \left[\frac{\partial^2 A^{\lambda\mu}}{\partial x^\alpha \partial x^\beta}\right], p_i^0.$$

These are bounded and continuous functions, within W, of the three variables $\lambda_1, \lambda_2, \lambda_3$.

14 Derivatives of the functions σ_s^r

We will use in the study of partial derivatives with respect to x^i of the functions σ_s^r, of the partial derivatives of polynomials considered in the remarks of Sec. 12: $\lambda_1^2 D, \lambda_1^2 p_j^0 D_i^j$ and $\lambda_1(\delta_j^h - p_j^0 p_h^0) D_i^j$ are polynomials of the functions $X(p_i, y_i^j), [A^{\lambda\mu}], p_i^0$, *homogeneous of degree 2, 2 and 1, respectively, with respect to the* y_i^j. The previous results and the identity (10.1) show then that the quantities

$$\frac{1}{\lambda_1}\frac{\partial}{\partial x^i}(\lambda_1^2 D),\ \frac{1}{\lambda_1}\frac{\partial}{\partial x^i}(\lambda_1^2 p_j^0 D_i^j),$$
$$\frac{\partial}{\partial x^i}(\lambda_1(\delta_j^h - p_j^0 p_h^0) D_i^j)$$

are rational fractions with denominator D of the functions

$$X(p_i, y_i^j, z_i^j), \tilde{X}\left(\frac{y_i^j}{\lambda_1}, \frac{y_{ih}^j}{\lambda_1}\right), [A^{\lambda\mu}], \left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right], p_i^0.$$

They are therefore continues and bounded functions of $\lambda_1, \lambda_2, \lambda_3$ in W.

In the study of second partial derivatives of the function σ with respect to the x^i we will use the second partial derivatives $\frac{\partial^2(\lambda_1^2 D)}{\partial x^i \partial x^j}$. Let us first remark that the first-order partial derivatives of $\lambda_1^2 D$ can be written

$$\frac{\partial(\lambda_1^2 D)}{\partial x^i} = \frac{P_1}{\lambda_1^2 D},$$

where P_1 is a polynomial of the functions

$$X(p_i, y_i^j, z_i^j, y_{ih}^j), [A^{\lambda\mu}], \left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right], p_i^0$$

whose terms are of the third degree at least with respect to the set of functions y_i^j, y_{ih}^j. As a matter of fact, the partial derivatives $\frac{\partial p_h}{\partial x^i}$ and $\frac{\partial p_h^0}{\partial x^i}$ can be put (by multiplying denominator and numerator of the second members of the equations by λ_1^2) in form of rational fractions with denominator $\lambda_1^2 D$ and whose numerators are polynomials of the functions

$$X(p_i, y_i^j, z_i^j), [A^{\lambda\mu}], \left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right], p_i^0$$

whose terms are of first degree at least with respect to the y_i^j, and the partial derivatives $\frac{\partial y_h^k}{\partial x^i}$ can be put in form of rational fractions with denominator $\lambda_1^2 D$ and whose numerators are polynomials of the functions

$$X(p_i, y_i^j, z_i^j, y_{hk}^j), [A^{\lambda\mu}], \left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right], p_i^0$$

homogeneous of second degree with respect to the set of functions y_i^0, y_{hk}^i. The polynomial $\lambda_1^2 D$ being homogeneous of first degree with respect to the y_i^j, its first partial derivatives have for sure the desired form

Let us then consider the second partial derivatives:

$$\frac{\partial^2(\lambda_1^2 D)}{\partial x^i \partial x^j} = \frac{1}{\lambda_1^2 D}\frac{\partial p_1}{\partial x^i} = \frac{P_1}{(\lambda_1^2 D)^2}\frac{\partial(\lambda_1^2 D)}{\partial x^i}.$$

It turns out from the form of the polynomial P_1 and from the results of Sec. 12 that:

(1) $\frac{P_1}{\lambda_1^3}$ is a polynomial of the functions

$$X(p_i, y_i^j, z_i^j, y_{hk}^j), X\left(\frac{y_i^j}{\lambda_1}, -\frac{y_{ih}^j}{\lambda_1}\right), [A^{\lambda\mu}], \left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right], p_i^0.$$

(2) $\frac{1}{\lambda_1^2}\frac{\partial P_1}{\partial x^i}$ is a rational fraction with denominator D of the functions

$$X(p_i, y_i^j, z_i^j, y_{hk}^j, z_{ih}^j, y_{ihk}^j), \tilde{X}\left(\frac{y_i^j}{\lambda_1}, \frac{y_{ih}^j}{\lambda_1}, \frac{y_{ihk}^j}{\lambda_1}\right), [A^{\lambda\mu}], ..., \left[\frac{\partial^2 A^{\lambda\mu}}{\partial x^\alpha \partial x^\beta}\right], p_i^0.$$

The derivatives $\frac{\partial^2(\lambda_1^2 D)}{\partial x^i \partial x^j}$ are therefore rational fractions with denominator D^3 of the functions we have just listed.

15 Study of σ and of its derivatives

(1^o) The auxiliary function σ has been defined by $\sigma = \left|\frac{\sin\lambda_2}{\triangle}\right|^{\frac{1}{2}}$. We have therefore, by virtue of the equality (12.2),

$$\sigma = \frac{1}{|\lambda_1^2 D|^{\frac{1}{2}}}.$$

One deduces that, in the domain W, the function $\sigma\lambda_1 = \frac{1}{|D|^{\frac{1}{2}}}$ is the square root of a rational fraction, bounded and nonvanishing, of the function

$$X, \tilde{X}, [A^{\lambda\mu}], p_i^0;$$

it is a continuous and bounded function of the three variables λ_i, whose value for $\lambda_1 = D$ is

$$\lim_{\lambda_1 \to 0} \sigma\lambda_1 = 1. \tag{15.1}$$

(2^o) The first partial derivatives of σ with respect to the x^i are

$$\frac{\partial\sigma}{\partial x^i} = \frac{\sigma}{2}\frac{1}{\lambda_1^2 D}\frac{\partial(\lambda_1^2 D)}{\partial x^i}.$$

One concludes that, in the domain W, the function

$$\lambda_1^2 \frac{\partial \sigma}{\partial x^i} = -\frac{\sigma}{2} \frac{\lambda_1}{D} \frac{1}{\lambda_1} \frac{\partial(\lambda_1^2 D)}{\partial x^i}$$

is the product of the square root of a nonvanishing bounded rational fraction with a bounded rational fraction of the functions $X, \tilde{X}, [A^{\lambda\mu}], \left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right], p_i^0$. It is a continuous and bounded function of $\lambda_1, \lambda_2, \lambda_3$ of which we are going to compute the value for $\lambda_1 = 0$.

The identities $\frac{\partial \sigma}{\partial \lambda_1} = T^i \frac{\partial \sigma}{\partial x^i}$ and $\frac{\partial \sigma}{\partial p_h^0} = \frac{\partial \sigma}{\partial x^i} y_h^i$ show that the functions $\lambda_1^2 \frac{\partial \sigma}{\partial \lambda_1}$ and $\lambda_1 \frac{\partial \sigma}{\partial p_h^0}$ are continuous and bounded in W. We can therefore, on the one hand differentiate the equality (15.1) with respect to p_h^0, we find

$$\lim_{\lambda_1 \to 0} \lambda_1 \frac{\partial \sigma}{\partial p_h^0} = 0,$$

on the other hand we can write

$$\frac{\partial(\sigma \lambda_1^2)}{\partial \lambda_1} = 2\lambda_1 \sigma + \lambda_1^2 \frac{\partial \sigma}{\partial \lambda_1}$$

and

$$\lim_{\lambda_1 \to 0} \frac{\partial(\sigma \lambda_1^2)}{\partial \lambda_1} = \lim_{\lambda_1 \to 0} \lambda_1 \sigma,$$

from which

$$\lim_{\lambda_1 \to 0} \lambda_1^2 \frac{\partial \sigma}{\partial \lambda_1} = -\lim_{\lambda_1 \to 0} \lambda_1 \sigma = -1.$$

In order to compute the value for $\lambda_1 = 0$ of the function $\lambda_1^2 \frac{\partial \sigma}{\partial x^i}$ we shall use the identity

$$\lambda_1^2 \frac{\partial \sigma}{\partial x^i} = \lambda_1^2 \frac{\partial \sigma}{\partial \lambda_1} \frac{\triangle_1^i}{\triangle} + \lambda_1 \frac{\partial \sigma}{\partial p_h^0} \lambda_1 \frac{\partial p_h^0}{\partial \lambda_u} \frac{\triangle_u^i}{\triangle},$$

from which, in light of the previous results (Sec. 12),

$$\lim_{\lambda_1 \to 0} \lambda_1^2 \frac{\partial \sigma}{\partial x^i} = p_i^0. \tag{15.2}$$

(3^o) The second partial derivatives of σ with respect to the x^i are

$$\frac{\partial^2 \sigma}{\partial x^i \partial x^j} = -\frac{\sigma}{2} \frac{1}{\lambda_1^2 D} \frac{\partial^2(\lambda_1^2 D)}{\partial x^i \partial x^j} - \frac{1}{2\lambda_1^2 D} \frac{\partial \sigma}{\partial x^j} \frac{\partial(\lambda_1^2 D)}{\partial x^i} + \frac{\sigma}{2(\lambda_1^2 D)^2} \frac{\partial(\lambda_1^2 D)}{\partial x^i} \frac{\partial(\lambda_1^2 D)}{\partial x^j}.$$

(a) It is easily seen that in the domain W the function $\lambda_1^3 \frac{\partial^2 \sigma}{\partial x^i \partial x^j}$ is the product of the square root of a nonvanishing bounded rational fraction with a bounded rational fraction (having denominator D^4) of the functions

$$X, \tilde{X}, [A^{\lambda\mu}], \left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right], \left[\frac{\partial^2 A^{\lambda\mu}}{\partial x^\alpha \partial x^\beta}\right], p_i^0.$$

It is a continuous and bounded function of the three variables λ_i. We are going to compute the value for $\lambda_1 = 0$ of the function $\lambda_1^3 \sum_{i=0}^3 \frac{\partial^2 \sigma}{\partial x^{i\,2}}$ whi, only, we will need: the second derivatives of σ do not occur actually in the fundamental equations except for the quantity $[A^{ij}] \frac{\partial^2 \sigma}{\partial x^i \partial x^j}$ and one has

$$\lim_{\lambda_1 \to 0} [A^{ij}] \lambda_1^3 \frac{\partial^2 \sigma}{\partial x^i \partial x^j} = \lim_{\lambda_1 \to 0} \lambda_1^3 \sum_{i=1}^3 \frac{\partial^2 \sigma}{\partial x^{i\,2}}.$$

We will evaluate this limit as the limit (15.2). We shall find on the one hand, by differentiating the equality (15.2),

$$\lim_{\lambda_1 \to 0} \lambda_1^2 \frac{\partial}{\partial p_h^0} \left(\frac{\partial \sigma}{\partial x^i}\right) = \delta_h^i,$$

on the other hand

$$\frac{\partial}{\partial\lambda_1}\left(\lambda_1^3\frac{\partial\sigma}{\partial x^i}\right)=3\lambda_1^2\frac{\partial\sigma}{\partial x^i}+\lambda_1^3\frac{\partial}{\partial\lambda_1}\left(\frac{\partial\sigma}{\partial x^i}\right),$$

from which

$$\lim_{\lambda_1\to0}\lambda_1^3\frac{\partial}{\partial\lambda_1}\left(\frac{\partial\sigma}{\partial x^i}\right)=\lim_{\lambda_1\to0}\left(-2\lambda_1^2\frac{\partial\sigma}{\partial x^i}\right)=-p_i^0.$$

We find therefore, by using the identity

$$\sum_{i=1}^{3}\lambda_1^3\frac{\partial^2\sigma}{\partial {x^i}^2}=\lambda_1^3\frac{\partial}{\partial\lambda_1}\left(\frac{\partial\sigma}{\partial x^i}\right)\frac{\triangle_i^1}{\triangle}+\lambda_1^3\frac{\partial}{\partial p_h^0}\left(\frac{\partial\sigma}{\partial x^i}\right)\frac{\partial p_h^0}{\partial\lambda_u}\frac{\triangle_i^u}{\triangle}$$

and the results of previous paragraphs, that

$$\lim_{\lambda_1\to0}\sum_{i=1}^{3}\lambda_1^3\frac{\partial^2\sigma}{\partial {x^i}^2}=0.$$

Let us show that the function $\lambda_1^2[A^{ij}]\frac{\partial^2\sigma}{\partial x^i\partial x^j}$ is a continuous and bounded function of the three variables λ_i, in the neighbourhood of $\lambda_1=0$ (which will make it possible for us to prove that the quantity under the sign $\int\int\int$ (6.1) is bounded in W).

We have seen that $\lambda_1^3\frac{\partial^2\sigma}{\partial x^i\partial x^j}[A^{ij}]$ is the product of a square root of a nonvanishing bounded rational fraction $\left(\frac{1}{D}\right)$ with a rational fraction having denominator D^4, whose numerator, polynomial of the functions

$$X,\tilde{X},[A^{\lambda\mu}],\left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right],\left[\frac{\partial^2A^{\lambda\mu}}{\partial x^\alpha\partial x^\beta}\right],p_i^0,$$

vanishes for the values of these functions corresponding to $\lambda_1=0$. We have

$$\lambda_1^3[A^{ij}]\frac{\partial^2\sigma}{\partial x^i\partial x^j}=\frac{P\left(X,\tilde{X},[A^{\lambda\mu}],\left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right],\left[\frac{\partial^2A^{\lambda\mu}}{\partial x^\alpha\partial x^\beta}\right],p_i^0\right)}{D^4}\frac{1}{|D|^{\frac12}}$$

with

$$P_0=P\left(X_0,\tilde{X}_0,\pm\delta_\lambda^\mu,\left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right]_0,\left[\frac{\partial^2A^{\lambda\mu}}{\partial x^\alpha\partial x^\beta}\right]_0,p_i^0\right)=0.$$

We then write:

$$\lambda_1^3[A^{ij}]\frac{\partial^2\sigma}{\partial x^i\partial x^j}=\frac{P-P_0}{D^4}\frac{1}{|D|^{\frac12}}.\tag{15.3}$$

By applying the Taylor formula (for P) one sees that the quantity (15.3) is a polynomial of the functions $X-X_0,\tilde{X}-\tilde{X}_0,A^{\lambda\mu}\pm\delta_\lambda^\mu$ ..., whose terms are of first degree at least with respect to the set of these functions.

To show that $\lambda_1^2[A^{ij}]\frac{\partial^2\sigma}{\partial x^i\partial x^j}$ is a continuous and bounded function of $\lambda_1,\lambda_2,\lambda_3$ in the domain W, it is enough to show that the same holds for the functions

$$\frac{X-X_0}{\lambda_1},\frac{\tilde{X}-\tilde{X}_0}{\lambda_1},\frac{[A^{\lambda\mu}]-\delta_\mu^\lambda}{\lambda_1},...,\frac{\left[\frac{\partial^2A^{\lambda\mu}}{\partial x^\alpha\partial x^\beta}\right]-\left[\frac{\partial^2A^{\lambda\mu}}{\partial x^\alpha\partial x^\beta}\right]_0}{\lambda_1}.$$

The functions X verify

$$X=\int_0^{\lambda_1}E(X)d\lambda_1+X_0,$$

$\frac{X-X_0}{\lambda_1}$ is therefore a continuous and bounded function of the λ_i in V:

$$|X-X_0|\le\lambda_1M.\tag{15.4}$$

The coefficients $A^{\lambda\mu}$ possessing in (D) partial derivatives continuous and bounded up to the fourth order with respect to the x^α, the x^α fulfilling the inequalities (15.4), we see that

$$[A^{\lambda\mu}] \pm \delta^\mu_\lambda \leq \lambda_1 A, ..., \left[\frac{\partial^3 A^{\lambda\mu}}{\partial x^\alpha \partial x^\beta \partial x^\gamma}\right] - \left[\frac{\partial^3 A^{\lambda\mu}}{\partial x^\alpha \partial x^\beta \partial x^\gamma}\right]_0 \leq \lambda_1 A. \tag{15.5}$$

Let us consider $\frac{X-X_0}{\lambda_1}$. The corresponding X functions are $y^j_i, y^j_{ih}, y^j_{ihk}$ which verify the equation

$$X = \int_0^{\lambda_1} E(X) d\lambda_1,$$

$E(X)$ being a polynomial of the functions X, of the $A^{\lambda\mu}$ and of their partial derivatives up to the third order

$$\left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right], ..., \frac{\partial^3 A^{\lambda\mu}}{\partial x^\alpha \partial x^\beta \partial x^\gamma}.$$

We have

$$\tilde{X} - \tilde{X}_0 = \frac{\int_0^{\lambda_1}(E(X) - E(X)_0) d\lambda_1}{\lambda_1^2}.$$

The Taylor formula applied to the polynomial E shows that $E(X) - E(X)_0$ is a polynomial of the functions

$$X_0, \delta^\mu_\lambda, ..., \left[\frac{\partial^3 A^{\lambda\mu}}{\partial x^\alpha \partial x^\beta \partial x^\gamma}\right]_0$$

and of the functions

$$X - X_0, [A^{\lambda\mu}] - \delta^\mu_\lambda, ..., \left(\left[\frac{\partial^3 A^{\lambda\mu}}{\partial x^\alpha \partial x^\beta \partial x^\gamma}\right] - \left[\frac{\partial^3 A^{\lambda\mu}}{\partial x^\alpha \partial x^\beta \partial x^\gamma}\right]_0\right)$$

whose terms are of first degree at least with respect to this last set of terms.

All these functions being bounded in V and satisfying (15.4) and (15.5) we see easily that $\frac{\tilde{X}-\tilde{X}_0}{\lambda_1}$ is continuous and bounded in V.

The function $\lambda_1^2 [A^{ij}] \frac{\partial^2 \sigma}{\partial x^i \partial x^j}$ is therefore continuous and bounded in W.

16 Derivatives of the ω^r_s

We are going to prove that the first and second partial derivatives of the ω^r_s with respect to the x^i are, as σ and its partial derivatives, simple algebraic functions of the functions X and Ω, $\tilde{X}$ and $\tilde{\Omega}$, and of the values on the conoid Σ_0 of the coefficients of the given equations and of their partial derivatives.

(1^o) The first partial derivatives of the ω^r_s with respect to the x^i are expressed as functions of their partial derivatives with respect to the λ_i

$$\frac{\partial \omega^r_s}{\partial x^i} = \frac{\partial \omega^r_s}{\partial \lambda_j} \frac{\triangle^j_i}{\triangle},$$

therefore

$$\frac{\partial \omega^r_s}{\partial x^i} = (Q^r_t \omega^t_s + Q\omega^r_s)\frac{P^0_i D^j_i}{D} + \frac{\omega^r_{sh}}{\lambda_1}\frac{(\delta^h_j - P^0_j P^0_h) D^j_i}{D}. \tag{16.1}$$

The first partial derivatives of the ω^r_s with respect to the x^i are therefore rational fractions with denominator D of the functions

$$X(P_i, y^j_i), \Omega(\omega^r_s), \tilde{X}\left(\frac{y^j_i}{\lambda_1}\right), \tilde{\Omega}\left(\frac{\omega^r_{sh}}{\lambda_1}\right), [A^{\lambda\mu}], \left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right], [B^{s\lambda}] \text{ and } P^0_i.$$

These are continuous and bounded functions in W.

(2^o) We will compute the second partial derivatives of the ω^r_s with respect to the x^i by writing $\frac{\partial \omega^r_s}{\partial x^i}$ in the form $\frac{\partial \omega^r_s}{\partial x^i} = \frac{P_2}{\lambda_1^2 D}$.

The equality (16.1) and the remarks of Sec. 12 show that P_2 is a homogeneous polynomial of second degree with respect to the set of functions y^j_i, ω^r_s. We have, by differentiating the previous equality,

$$\frac{\partial^2 \omega^r_s}{\partial x^i \partial x^j} = \frac{1}{\lambda_1^2 D}\frac{\partial P_2}{\partial x^j} - \frac{P_2}{(\lambda_1^2 D)^2}\frac{\partial(\lambda_1^2 D)}{\partial x^i}.$$

These functions $\lambda_1 \frac{\partial^2 \omega^r_s}{\partial x^i \partial x^j}$ are rational fractions with denominator D^3 of the functions

$$X, \Omega, \tilde{X}, \tilde{\Omega}, [A^{\lambda\mu}], \left[\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right], \left[\frac{\partial^2 A^{\lambda\mu}}{\partial x^\alpha \partial x^\beta}\right][B^{s\lambda}_r], \left[\frac{\partial B^{s\lambda}_r}{\partial x^\alpha}\right].$$

(The results of Sec. 12 make it possible actually to prove that $\frac{P_2}{\lambda_1^2}$ and $\frac{1}{\lambda_1}\frac{\partial P_2}{\partial x^j}$ are a polynomial and a rational fraction, respectively, with denominator D, of these functions.) These are therefore continuous and bounded functions in W.

17 C. Kirchhoff formulas

We can now study in more precise way the fundamental equations (6.1) and look for their limit as η tends to zero.

These equations read as:

$$\begin{aligned} &\int\int_V \int ([u_r]L^r_s + \sigma^r_s[f_r])dx^1\,dx^2\,dx^3 \\ +\quad &\int\int_{S_0} E^i_s \cos(n,x^i)dS = \int\int_{S_\eta} E^i_s \cos(n,x^i)dS. \end{aligned} \tag{17.1}$$

Integral relations involving the parameter λ_i. We have seen that the functional determinant $D = \frac{D(x^i)}{D(\lambda_j)}$ is equal to -1 for $\lambda_1 = 0$. The correspondence between the parameters x^i and λ_j is therefore surjective in a neighbourhood of the vertex M_0 of Σ_0. One derives from this that the correspondence between the parameters x^i and λ_j is one-to-one in a domain $(\Lambda)_\eta$ defined by

$$\eta \leq \lambda_1 \leq \varepsilon_3,\ 0 \leq \lambda_2 \leq \pi,\ 0 \leq \lambda_3 \leq 2\pi,$$

where ε_3 is a given number and where η is arbitrarily small.

To the domain $(\Lambda)_\eta$ of variations of the λ_i parameters there corresponds, in a one-to-one[5] way, a domain W_η of Σ_0. We shall then assume that the coordinate x^4_0 of the vertex M_0 of Σ_0 is sufficiently small to ensure that the domain $V_\eta \subset V$, previously considered, is interior to the domains W and W_η. We can, under these conditions, compute the integrals by means of the parameters λ_i, the integrals that we are going to obtain being convergent.

18 Calculation of the area and volume elements

First, we have

$$dV = dx^1\,dx^2\,dx^3 = d\lambda_1\,d\lambda_2\,d\lambda_3.$$

Let us compute now dS and $\cos(n, x^i)$.

[5] Because the correspondence between $(x^4, \lambda_2, \lambda_3)$ and $(\lambda_1, \lambda_2, \lambda_3)$ is one-to-one.

The surfaces S_0 and S_η are $x^4 = c^{\text{te}}$ surfaces drawn on the characteristic conoid Σ_0. They therefore satisfy the differential relation

$$p_i \, dx^i = 0,$$

from which one deduces

$$\cos(n, x^i) = \frac{p_i}{(\sum p_i^2)^{\frac{1}{2}}}.$$

In order to evaluate dS we shall write a second expression of the volume element dV in which the surfaces $S(x^4 = c^{\text{te}})$ and the bicharacteristics (where only λ_1 is varying) come into play

$$dV = \cos\nu \, |T|^{\frac{1}{2}} d\lambda_1 \, dS,$$

where $|T|^{\frac{1}{2}} d\lambda_1$ denotes the length element of the bicharacteristic, and ν is the angle formed by the bicharacteristic with the normal to the surface S at the point considered.

A system of directional parameters of the tangent to the bicharacteristic being

$$T^h = [A^{hj}]p_j + [A^{h4}],$$

we have

$$\cos\nu \, |T|^{\frac{1}{2}} = \left\{[A^{hj}]p_j + [A^{h4}]\right\} \cos(n, x^h),$$

from which, by comparing the two expressions of dV,

$$\cos(n, x^i) dS = \frac{\triangle p_i d\lambda_2 \, d\lambda_3}{[A^{hj}]p_j p_h + [A^{h4}]p_h} = \frac{-\triangle p_i}{[A^{44}] + [A^{j4}]p_j} d\lambda_2 \, d\lambda_3.$$

19 Limit as $\eta \to 0$ of the integral relations

The integral relations (17.1) read, in terms of the λ_i parameters, as

$$\begin{aligned} & \int\int_{V_\eta}\int ([u_r]L_s^r + \sigma_s^r[f_r]) d\lambda_1 \, d\lambda_2 \, d\lambda_3 \\ - & \int_0^{2\pi}\int_0^{\pi} \frac{E_s^i \triangle p_i}{[A^{44}] + [A^{i4}]p_i} d\lambda_{2|x^4=0} \, d\lambda_3 \\ = & -\int_0^{2\pi}\int_0^{\pi} \left\{\frac{E_s^i \triangle p_i}{[A^{44}] + [A^{i4}]p_i}\right\}_{x^4 = {x_0}^4 - \eta} d\lambda_2 \, d\lambda_3. \end{aligned} \tag{19.1}$$

The previous results prove that the quantities to be integrated are continuous and bounded functions of the variables λ_i. They read actually as:

$$\lambda_1^2 \left\{[u_r]L_s^r + \sigma_s^r[f_r]\right\} \frac{\triangle}{\lambda_1^2} \text{ and } \lambda_1^2 E_s^i \frac{\triangle}{\lambda_1^2} \frac{p_i}{T^4}.$$

E_s^i and L_s^r being given by the equalities (5.2), the quantities considered are continuous and bounded in W if the functions u_r and $\frac{\partial u_r}{\partial x^\alpha}$ are continuous and bounded in D.

The two members of equations (19.1) tend therefore towards a finite limite when η tends to zero. The triple integral tends to a finite limit, equal to the value of this integral taken over the portion V_0 of hypersurface of the conoid Σ_0 in between the vertex M_0 and the initial surface $x^4 = 0$ (because this integral is convergent). Let us evaluate the limit of the double integral of the second member. The results of Sec. 15 show that all terms of the quantity $\lambda_1^2 E_s^i$ tend uniformly to zero with λ_1, exception being made for the term

$$-\lambda_1^2 [u_r][A^{ij}]\omega_s^r \frac{\partial \sigma}{\partial x_j},$$

whose limit for $\lambda_1 = 0$ is

$$[u_r(x_0^\alpha)]\delta_i^j\delta_s^r p_j^0 = u_s(x_0^\alpha)p_i^0.$$

From which:

$$\lim_{\lambda_1\to 0}\frac{E_s^i \triangle p_i}{[A^{44}]+[A^{i4}]p_i} = -u_s(x_0^\alpha)\sin\lambda_2.$$

The second member of equations (19.1) tends therefore, when η tends to zero, to the limit

$$\int_0^{2\pi}\int_0^{\pi} u_s(x_0^\alpha)\sin\lambda_2\, d\lambda_2\, d\lambda_3 = 4\pi u_s(x_0^\alpha).$$

20 Kirchhoff formulas

We arrive in such a way to the following formulas:

$$\begin{aligned} 4\pi u_s(x_0^\alpha) &= \int\int_{V_\eta}\int([u_r]L_s^r + \sigma_s^r[f_r])\triangle\ d\lambda_1\, d\lambda_2\, d\lambda_3 \\ &+ \int_0^{2\pi}\int_0^{\pi}\left\{\frac{E_s^i\triangle p_i}{T^4}\right\}_{x^4=0} d\lambda_2\, d\lambda_3. \end{aligned} \tag{20.1}$$

In order to compute the second member of these Kirchhoff formulas it will be convenient to take for parameters, on the hypersurface of the conoid Σ_0, the three independent variables $x^4, \lambda_2, \lambda_3$.

The equations (20.1) then read, the limits of integration intervals being evident:

$$\begin{aligned} 4\pi u_s(x_j) &= \int_{x_0^4}^{0}\int_0^{2\pi}\int_0^{\pi}([u_r]L_s^r + \sigma_s^r[f_r])\frac{\triangle}{T^4}dx^4 d\lambda_2 d\lambda_3 \\ &+ \int_0^{2\pi}\int_0^{\pi}\left\{\frac{E_s^i\triangle p_i}{T^4}\right\}_{x^4=0} d\lambda_2 d\lambda_3. \end{aligned} \tag{20.2}$$

The quantity under sign of triple integral is expressed by means of the functions $[u]$ and of the functions $X(\lambda_1,\lambda_2,\lambda_3)$ and $\Omega(\lambda_1,\lambda_2,\lambda_3)$, solutions of the integral equations (1.2) and (8.1).

We shall obtain the expression of the X and Ω as functions of the new variables $x^4, \lambda_2, \lambda_3$ by replacing λ_1 with its value defined by the equation (2.2), function of the $x^4, \lambda_2, \lambda_3$.

Let us point out that these functions satisfy the integral equations

$$X(x^4,\lambda_2,\lambda_3) = \int_{x_0^4}^{x^4}\frac{E(X)}{T^4}dx^4 + X_0(x_0^4,\lambda_2,\lambda_3)$$

$$\Omega(x^4,\lambda_2,\lambda_3) = \int_{x_0^4}^{x^4}\frac{F(X,\Omega)}{T^4}dx^4 + \Omega_0(x_0^4,\lambda_2,\lambda_3).$$

The quantity under sign of double integral is expressed by means of the values for $x^4 = 0$ of the functions $[u]$ and $\left[\frac{\partial u}{\partial x^\alpha}\right]$ (Cauchy data) and of the values for $x^4 = 0$ of the functions X and Ω.

21 D. Summary of the results

We shall consider a system of linear, second-order partial differential equations in four variables, of the type

$$A^{\lambda\mu}\frac{\partial^2 u_r}{\partial x^\lambda\partial x^\mu} + B_s^{r\lambda}\frac{\partial u_s}{\partial x^\lambda} + f_r = 0. \tag{E}$$

Assumptions

(1^o) At the point M_0 of coordinates x_0^α the coefficients $A^{\lambda\mu}$ take the following values:

$$A_0^{44} = 1,\ A_0^{i4} = 0,\ A_0^{ij} = -\delta_i^j.$$

(2^o) The coefficients $A^{\lambda\mu}$ and $B_s^{r\lambda}$ have partial derivatives with respect to the x^α, of orders four and two, respectively, continuous and bounded in a domain $D : |x^i - \tilde{x}^i| \le d, |x^4| \le \varepsilon$. The coefficients f_r are continuous and bounded.

(3^o) The partial derivatives of the $A^{\lambda\mu}$ and $B_s^{r\lambda}$ of orders four and two, respectively, satisfy in D some Lipschitz conditions.

Conclusion. Every solution of the equations (E) continuous, bounded and with first partial derivatives continuous and bounded in D verifies the integral relations (20.2) if the coordinates x_0^α of M_0 satisfy inequalities of the form

$$|x_0^4| \le \varepsilon_0,\ |x_0^i - \tilde{x}^i| \le d,$$

defining a domain $D_0 \subset D$.

22 II. Transformation of variables

We are going to try establishing formulas analogous to (20.2), verified by the solutions of the given equations (E) at every point of a domain D_0 of spacetime, where the values of coefficients will be restricted uniquely by the requirement of having to verify some conditions of normal hyperbolicity and differentiability.

Let us therefore consider the system (E) of equations

$$A^{\lambda\mu}\frac{\partial^2 u_s}{\partial x^\lambda \partial x^\mu} + B_s^{r\lambda}\frac{\partial u_r}{\partial x^\lambda} + f_s = 0.$$

We assume that in the spacetime domain D, defined by

$$|x^4| \le \varepsilon,\ |x^i - \tilde{x}^i| \le d,$$

where the three $\tilde{x}^i$ are given numbers, the equations (E) are of the normal hyperbolic type, i.e.

$$A^{44} > 0,\ \text{the quadratic form } A^{ij}X_iX_j \text{ negative definite.}$$

At every point $M_0(x_j)$ of the domain D one can associate to the values $A_0^{\lambda\mu} = A^{\lambda\mu}(x_0^\alpha)$ of the coefficients A a system of real numbers $a_0^{\alpha\beta}$, algebraic functions, defined and indefinitely differentiable of the $A_0^{\lambda\mu}$, satisfying the identity

$$A_0^{\lambda\mu}X_\lambda X_\mu = (a_0^{4\alpha}X_\alpha)^2 - (a_0^{i\alpha}X_\alpha)^2.$$

We shall denote by $a^0_{\alpha\beta}$ the quotient by the determinant a_0 of elements $a_0^{\alpha\beta}$ of the minor relative to the element $a_0^{\alpha\beta}$ of this determinant. The quantities $a^0_{\alpha\beta}$ are, like $a_0^{\alpha\beta}$, algebraic functions defined and indefinitely differentiable of the $A_0^{\lambda\mu}$ in D. (The square of the determinant a_0, being equal to the absolute value A of the determinant having elements $A^{\lambda\mu}, a_0$, is different from zero in D.)

Let us perform the linear change of variables

$$y_\alpha = a^0_{\alpha\beta}\, x^\beta.$$

The partial derivatives of the unknown functions u_s are covariant in such a change of variables, hence the equations (E) read as

$$A^{*\alpha\beta}\frac{\partial^2 u_s}{\partial y^\alpha \partial y^\beta} + B_s^{*r\alpha}\frac{\partial u_s}{\partial y^\alpha} + f_s = 0, \tag{22.1}$$

with
$$A^{*\alpha\beta} = A^{\lambda\mu} a^0_{\alpha\lambda} a^0_{\beta\mu},\ B^{*r\alpha}_s = B^{r\lambda}_s a^0_{\alpha\lambda}. \tag{22.2}$$

The coefficients of equations (22.1) take at the point M_0 the values (1.4). As a matter of fact:

$$A^{*\alpha\beta}_0 = A^{\lambda\mu}_0 a^0_{\alpha\lambda} a^0_{\beta\mu} = -a^{\gamma\lambda}_0 a^{\gamma\mu}_0 a^0_{\alpha\lambda} a^0_{\beta\mu} + 2a^{4\lambda}_0 a^{4\mu}_0 a^0_{\alpha\lambda} a^0_{\beta\mu} = -\delta^\beta_\alpha + 2\delta^4_\alpha \delta^4_\beta,$$

hence one has
$$A^{*44} = 1,\ A^{*i4} = 0,\ A^{*ij} = -\delta^j_i.$$

We can apply to the equations (E), written in the form (22.1), in the variables y^α and for the corresponding point M_0, the results of part I. Let us first point out that the integration parameters so introduced will be $y^4, \lambda_2, \lambda_3$ but that, the surface carrying the Cauchy data being always $x^4 \equiv a^{\alpha 4}_0 y^4 = 0$, the integration domains will be determined from M_0 and the intersection of this surface with the characteristic conoid with vertex M_0. We see that it will be convenient, in order to evaluate these integrals, to choose the variables y^α relative to a point M_0 whatsoever in such a way that the initial space section, $x^4 = 0$, is a hypersurface $y^4 = 0$. It will be enough for that purpose to choose the coefficients $a^{\alpha\beta}_0$ (which is legitimate) in such a way that one has $a^{i4}_0 = 0$. We shall then have

$$a^0_{4i} = 0,\ a^0_{44} = \frac{1}{a^{44}_0} = (A^{44}_0)^{-\frac{1}{2}} \text{ and } y_4 = a^0_{44} x^4,$$

where a^0_{44} is a bounded positive number.

23 Application of the results of part I

The application of the results of part I proves then the existence of a domain $D_0 \subset D$, defined by $|x^4_0| \le \varepsilon$ (which implies at every point $M_0 \in D_0, |y^4_0| \le \eta$) such that one can write at every point M_0 of D_0 a Kirchhoff formula whose first member is the value at M_0 of the unknown u_s, in terms of the quantities $y^\alpha_0 = a^0_{\alpha\beta} x^\beta_0$, and whose second member consists of a triple integral and of a double integral. The quantities to be integrated are expressed by means of the functions $X(y^4, \lambda_2, \lambda_3, y^\alpha_0)$ representing $(y^\alpha, p_i, y^j_i, z^j_i, ..., z^h_{kij})$ and $\Omega(y^4, \lambda_2, \lambda_3)$ $(\omega^r_s, ..., \omega^r_{sij})$, solutions of an equation of the kind
$$X = \int_{y^4_0}^{y^4} E^*(X) dy^4 + X_0,\ \Omega = \int_{y^4_0}^{y^4} F^*(X, \Omega) dy^4 + \Omega_0, \tag{23.1}$$
where the functions E^* and F^* are the functions E and F of Chapter I, but evaluated starting from the coefficients (22.2) and from their partial derivatives with respect to the y^α, and where Ω_0, X_0 denote the values for $y^4 = y^4_0$ of the corresponding functions Ω, X.

In order to obtain, under a simpler form, some integral equations holding in the whole domain D_0, we will take on the one hand as integration parameter, in place of y^4, x^4 (which is possible, a^0_{44} being at every point M_0 of D_0 a given positive number), we shall on the other hand replace those of the auxiliary unknown functions X which are the values (in terms of the three parameters) of the coordinates y^α of a point of the conoid Σ_0 of vertex M_0, with the values of the original coordinates x^α of a point of this conoid.

We shall replace for that purpose those of the integral equations which have in the first member y^α with their linear combinations of coefficients $a^{\alpha\beta}_0$ (bounded numbers), i.e. with the equations of the same kind
$$a^{\alpha\beta}_0 y^\beta = x^\alpha = \int_{x^4_0}^{x^4} a^{\alpha\beta}_0 \frac{T^{*\alpha\beta}}{T^{*4}} a^0_{44} dx^4 + x^\alpha_0,$$
and we will replace the quantities under integration signs of all our equations in terms of the x^α in place of the y^β by replacing in these equations the y^β with the linear combinations $a^0_{\alpha\beta} x^\alpha$ (the $a^0_{\alpha\beta}$ are bounded numbers).

The system of integral equations obtained in such a way has, for every point M_0 of the domain D, solutions as for the previous system, solutions which are of the form

$$X(x_0^\alpha, x^4, \lambda_2, \lambda_3).$$

24 Summary of results of chapter I

We consider a system of linear, second-order partial differential equations of the type

$$A^{\lambda\mu}\frac{\partial^2 u_s}{\partial x^\lambda \partial x^\mu} + B_s^{r\lambda}\frac{\partial u_r}{\partial x^\lambda} + f_s = 0. \qquad (E)$$

Assumptions

(1^o) In the domain D, defined by

$$|x^4| \leq \varepsilon,\ |x^i - \tilde{x}^i| \leq d,$$

the quadratic form $A^{\lambda\mu}X_\lambda X_\mu$ is of normal hyperbolic type:

$$A^{44} > 0,\ \text{the quadratic form } A^{ij}X_iX_j \text{ negative} - \text{definite}.$$

(2^o) The coefficients $A^{\lambda\mu}$ and $B_s^{r\lambda}$ have partial derivatives with respect to the x^α continuous and bounded, up to the orders four and two, respectively, in the domain D.

(3^o) The partial derivatives of the $A^{\lambda\mu}$ and $B_s^{r\lambda}$ of orders four and two, respectively, satisfy, within D, Lipschitz conditions.

Conclusion. Every solution of the equations (E), possessing in D first partial derivatives with respect to the x^α continuous and bounded, verifies, if x_0^α are the coordinates of a point M_0 of a domain D_0 defined by

$$|x_0^4| \leq \varepsilon_0 \leq \varepsilon,\ |x_0^i - \bar{x}^i| \leq d_0 \leq d,$$

some Kirchhoff formulas whose first members are the values at the point M_0 of the unknown functions u_s and whose second members consist of a triple integral (integration parameters $x^4, \lambda_2, \lambda_3$) and of a double integral (integration parameters λ_2, λ_3). The quantities to be integrated are expressed by means of functions $X(x_0^\alpha, x^4, \lambda_2, \lambda_3)$ and $\Omega(x_0^\alpha, x^4, \lambda_2, \lambda_3)$, themselves solutions of given integral equations (23.1), and of the unknown functions $[u_s]$; the quantity under the sign of double integral, which is taken for the zero value of the x^4 parameter, contains, besides the previous functions, the first partial derivatives of the unknown functions $\left[\frac{\partial u_s}{\partial x^\alpha}\right]$ (value over Σ_0 of the Cauchy data). We obtain in such a way a system of integral equations verified in D_0 from the solutions of the equations (E). We write this system in the following reduced form:

$$X = \int_{x_0^4}^{x^4} E\, dx^4 + X_0$$

$$4\pi U = \int_{x_0^4}^{0}\int_0^{2\pi}\int_0^{\pi} H\, dx^4\, d\lambda_2\, d\lambda_3 + \int_0^{2\pi}\int_0^{\pi} I\, d\lambda_2\, d\lambda_3.$$

CHAPTER II

1 Nonlinear equations

We consider a system (F) of n second-order partial differential equations, with n unknown functions and four variables, nonlinear of the following type:

$$A^{\lambda\mu}\frac{\partial^2 W_s}{\partial x^\lambda \partial x^\mu} + f_s = 0,\ s = 1, 2...4,\ \lambda, \mu = 1, 2...n.$$

The coefficients $A^{\lambda\mu}$ and f_s are given functions of the four variables x^α, the unknown functions W_s, and of their first derivatives $\frac{\partial W_s}{\partial x^\alpha}$.

The coefficients $A^{\lambda\mu}$ are the same for the n equations.

We point out that the calculations, made in the previous chapter for the linear equations (E), are valid for the nonlinear equations (F): it suffices to consider in these calculations the functions W_s as functions of the four variables x^α; the coefficients $A^{\lambda\mu}$ and f_s are then functions of these four variables and the previous calculations are valid, subject of course to considering in all formulas where there is occurrence of partial derivatives of the coefficients with respect to x^α these derivations as having been performed. One will have for example

$$\frac{\partial A^{\lambda\mu}}{\partial x^\alpha} = \frac{\partial A^{\lambda\mu}}{\partial W_s}\frac{\partial W_s}{\partial x^\alpha} + \frac{\partial A^{\lambda\mu}}{\partial(\partial W_s/\partial x^\beta)}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial W_s}{\partial x^\beta}\right).$$

By applying the previous results one would prove that, under certain assumptions, the solutions of equations (F) satisfy a system of integral equations analogous to (I), but whose second members contain, besides the auxiliary functions, the integration parameters and the unknown functions, the partial derivatives with respect to the x^α of these unknown functions (because the equations (I) involve the derivatives of the coefficients $A^{\lambda\mu}$, up to the fourth order, with respect to the x^α).

Thus, we do not apply directly to the equations (F) the results of previous chapters; but we are going to show that, by deriving suitably five times with respect to the variables x^α the given equations (F), and by applying to the obtained equations the results of chapter I, one obtains a system of integral equations whose first members are the unknown functions W_s, their partial derivatives with respect to the x^α up to the fifth order and some auxiliary functions X, Ω, and whose second members contain only these functions and the integration parameters.

2 Differentiation of the equations (F)

We assume that in a spacetime domain D, centred at the point $\overline{M}$ with coordinates $x^i, 0$ and defined by

$$|x^i - \overline{x}^i| \leq d, \ |x^4| \leq \varepsilon$$

and for values of the unknown functions W_s and their first partial derivatives satisfying

$$|W_s - \overline{W}_s| \leq l, \ \left|\frac{\partial W_s}{\partial x^\alpha} - \overline{\frac{\partial W_s}{\partial x^\alpha}}\right| \leq l \tag{2.1}$$

(where $\overline{W}_s$ and $\overline{\frac{\partial W_s}{\partial x^\alpha}}$ are the values of the functions W_s and $\frac{\partial W_s}{\partial x^\alpha}$ at the point $\overline{\mu}$) the coefficients $A^{\lambda\mu}$ and f_s admit of partial derivatives with respect to all their aguments up to the fifth order.

We shall then obtain, by differentiating five times the equations (F) with respect to the variables x^α, a system of N equations (N is the product by n of the number of derivatives of order five of a function of four variables) verified, in the domain D, by the solutions of equations (F) which satisfy the inequalities (2.1) and possess derivatives with respect to the x^α up to the seventh order.

Let us write this system of N equations. We set

$$\frac{\partial W_s}{\partial x^\alpha} = W_{s\alpha}, \ \frac{\partial^2 W_s}{\partial x^\alpha \partial x^\beta} = W_{s\alpha\beta}$$

and we denote by U_S the partial derivatives of order five of W_s

$$\frac{\partial^5 W_s}{\partial x^\alpha \partial x^\beta \partial x^\gamma \partial x^\delta \partial x^\varepsilon} = W_{s\alpha\beta\gamma\delta\varepsilon} = U_S, \ s = 1, 2, ...N.$$

Let us differentiate the given equations (F) with respect to any whatsoever of the variables x^α, we obtain n equations of the form

$$A^{\lambda\mu}\frac{\partial^2 W_{s\alpha}}{\partial x^\lambda \partial x^\mu} + \left\{\frac{\partial A^{\lambda\mu}}{\partial W_{r\nu}}W_{r\alpha} + \frac{\partial A^{\lambda\mu}}{\partial W_{r\nu}}\frac{\partial W_{r\nu}}{\partial x^\alpha} + \frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right\}\frac{\partial W_{s\mu}}{\partial x^\lambda}$$

$$+\frac{\partial f_s}{\partial W_r}W_{r\alpha}+\frac{\partial f_s}{\partial W_{r\nu}}\frac{\partial}{\partial x^\alpha}W_{r\nu}+\frac{\partial f_s}{\partial x^\alpha}=0.$$

Let us start again four times this procedure, we obtain the following system of N equations:

$$\begin{aligned}
& A^{\lambda\mu}\frac{\partial^2 W_{s\alpha\beta\gamma\delta\varepsilon}}{\partial x^\lambda \partial x^\mu}+\left\{\frac{\partial A^{\lambda\mu}}{\partial W_r}W_{r\alpha}+\frac{\partial A^{\lambda\mu}}{\partial W_{r\nu}}W_{r\nu\alpha}+\frac{\partial A^{\lambda\mu}}{\partial x^\alpha}\right\}\frac{\partial}{\partial x^\lambda}W_{s\beta\gamma\delta\varepsilon\mu} \\
+ \quad & \left\{\frac{\partial A^{\lambda\mu}}{\partial W_r}W_{r\beta}+\frac{\partial A^{\lambda\mu}}{\partial W_{r\nu}}W_{r\nu\beta}+\frac{\partial A^{\lambda\mu}}{\partial x^\beta}\right\}\frac{\partial}{\partial x^\lambda}W_{s\alpha\gamma\delta\varepsilon\mu}\cdots \\
+ \quad & \left\{\frac{\partial A^{\lambda\mu}}{\partial W_r}W_{r\varepsilon}+\frac{\partial A^{\lambda\mu}}{\partial W_{r\nu}}W_{r\nu\varepsilon}+\frac{\partial A^{\lambda\mu}}{\partial x^\varepsilon}\right\}\frac{\partial}{\partial x^\lambda}W_{s\alpha\beta\gamma\delta\mu}+\frac{\partial A^{\lambda\mu}}{\partial W_{r\nu}}\frac{\partial W_{r\nu\alpha\beta\gamma\delta}}{\partial x^\varepsilon} \\
+ \quad & \frac{\partial f_s}{\partial W_{r\nu}}\frac{\partial W_{r\nu\alpha\beta\gamma\delta}}{\partial x^\varepsilon}+F_S=0, \qquad (2.2)
\end{aligned}$$

where F_S is a function of the variables x^α, of the unknown functions W_s and of their partial derivatives up to the fifth order included, but not of the derivatives of higher order.

The fifth derivatives U_S of the functions W_s satisfy therefore, in the domain D and under the conditions specified, a system of N equations of the following type:

$$A^{\lambda\mu}\frac{\partial^2 U_S}{\partial x^\lambda \partial x^\mu}+B_S^{T\lambda}\frac{\partial U_T}{\partial x^\lambda}+F_S=0. \qquad (2.3)$$

The coefficients $A^{\lambda\mu}, B_S^{T\lambda}$ and F_S of these equations are polynomials of the coefficients $A^{\lambda\mu}$ and f_s of the given equations (F) and of their partial derivatives with respect to all arguments up to the fifth order, as well as of the unknown functions W_s and of their partial derivatives with respect to the x^α up to the fifth order. The coefficients $A^{\lambda\mu}$ depend only on the variables x^α, the unknown functions W_s and their first partial derivatives $W_{s\alpha}$, the coefficients $B_S^{T\lambda}$ depend only on the variables x^α, the unknown functions W_s and their first and second partial derivatives $W_{s\alpha}$ and $W_{s\alpha\beta}$.

3 Application to the equations obtained of the results of chapter I

We consider the equations (F) as a system of N linear equations of second order, with unknown functions U_S, and we apply to these equations the results of the previous chapter. We shall obtain a system of integral equations whose first members will be some auxiliary functions Ω, X and the unknown functions U_S; the quantities occurring under the integrals of the second members will be expressed by means of the auxiliary functions X, of the unknown functions U_S and of the value for $x^4=0$ of their first partial derivatives $\frac{\partial U_S}{\partial x^\alpha}$, of the integration parameters, as well as of the coefficients $A^{\lambda\mu}, B_S^{T\lambda}$ and F_S (viewed as functions of the x^α) and of their partial derivatives up to the orders four, three and zero. $A^{\lambda\mu}, B_S^{T\lambda}$ and F_S not involving the partial derivatives of the functions W_s except for the orders up to one, two and five, respectively, the second members of the integral equations considered will not contain, besides the auxiliary functions X, Ω, the functions U_S and the value for $x^4=0$ of their first derivatives, and the integration parameters, nothing but the unknown functions W_s and their partial derivatives up to the fifth order included.

Integral equations verified by the functions W_s and their derivatives

If the functions W_s and their partial derivatives up to the fifth order

$$W_{s\alpha}, W_{s\alpha\beta}, \ldots, W_{s\alpha\beta\gamma\delta\varepsilon}=U_S$$

are continuous and bounded in a spacetime domain D ($|x^i-\overline{x}^i|\leq d, |x^4|\leq\varepsilon$) they verify in this domain the integral relations

$$W_s(x^\alpha)=\int_0^{x^4}W_{s4}(x^i,t)dt+W_s(x^i,0)$$

....

$$W_{s\alpha\beta\gamma\delta}(x^\alpha) = \int_0^{x^4} W_{s\alpha\beta\gamma\delta 4}(x^i, t)dt + W_{s\alpha\beta\gamma\delta}(x^i, 0). \tag{3.1}$$

By adjoining to the system of integral equations, previously considered, the system (3.1), we shall then be able to obtain a system of integral equations, verified, under certain assumptions, by the solutions of the given equations (F), whose second members will only contain the functions occurring in the first members.

4 Cauchy data

We shall write this system of integral equations for the purpose of solving, for the given equations (F), the Cauchy problem: the search for solutions W_s of the equations (F) which take, as well as their first partial derivatives, some values given in a domain (d) of the initial hypersurface $x^4 = 0$:

$$W_s(x^i, 0) = \varphi_s(x^i),$$

$$\frac{\partial W_s}{\partial x^4}(x^i, 0) = \psi_s(x^i),$$

where φ_s and ψ_s are given functions of the three variables x^i in the domain (d). We will prove that, under the assumptions stated below, the data φ_s and ψ_s determine the values in (d) of the partial derivatives up to the sixth order of the solution W_s of the equations (E).

Assumptions

(1^o) In the domain (d), defined by

$$|x^i - \overline{x}^i| \leq d,$$

the functions φ_s and ψ_s admit of partial derivatives continuous and bounded with respect to the three variables x^i and satisfy the inequalities

$$|\varphi_s - \overline{\varphi}_s| \leq l_0 \leq l, \ |\psi_s - \overline{\psi}_s| \leq l_0 \leq l, \ \left| \frac{\partial \varphi_s}{\partial x^i} - \overline{\frac{\partial \varphi_s}{\partial x^i}} \right| \leq l_0 \leq l. \tag{4.1}$$

(2^o) In the domain (d) and for values of the functions

$$W_s = \varphi_s, \ \frac{\partial W_s}{\partial x^4} = \psi_s \text{ and } \frac{\partial W_s}{\partial x^i} = \frac{\partial \varphi_s}{\partial x^i},$$

satisfying the inequalities (4.1), the coefficients $A^{\lambda\mu}$ and f_s have partial derivatives continuous and bounded with respect to all their arguments, up to the fifth order.

(3^o) In the domain (d) and for the functions φ_s and ψ_s considered the coefficient A^{44} is different from zero.

It turns out actually from the first assumption that the values in (d) of partial derivatives up to the sixth order, corresponding to a differentiation at most with respect to x^4, of the solutions W_s of the assigned Cauchy problem are equal to the corresponding partial derivatives of the functions φ_s and ψ_s, and are continuous and bounded in (d).

The values in (d) of partial derivatives up to the sixth order of the functions W_s, corresponding to more than one derivative with respect to x^4, are expressed in terms of the previous ones, of the coefficients $A^{\lambda\mu}$ and f_s of the equations (F) and of their partial derivatives up to the fourth order. The third assumption shows actually that the equations (F) make it possible to evaluate, being given within (d) the values of the functions $W_s, W_{s\alpha}, W_{s\alpha i}$, the value in (d) of W_{s44}, from which one will deduce by differentiation the value in (d) of the partial derivatives corresponding to two differentiations with respect to x^4. The equations that are derivatives of the equations (F) with respect to the variables x^α (up to the fourth order) make it possible, in analogous manner, to evaluate in (d) the values of partial derivatives up to the sixth order of the functions W_s. It turns out from the three previous assumptions that all functions obtained are continuous and bounded in (d).

We shall set

$$W_{sj}(x^i, 0) = \varphi_{sj}(x^i),$$
$$U_S(x^i, 0) = \Phi_S(x^i),$$
$$\frac{\partial U_S}{\partial x^4}(x^i, 0) = \Psi_S(x^i).$$

5 Summary of the results of chapter II

We consider a system of n partial differential equations of second order, nonlinear, of the following kind:

$$A^{\lambda\mu}\frac{\partial^2 W_s}{\partial x^\lambda \partial x^\mu} + f_s = 0,$$

where $A^{\lambda\mu}$ and f_s are functions of the W_r and of their first partial derivatives, and of the four variables x^α.

We have seen that, under the assumptions of Sec. 2, the seven-times differentiable solutions of the equations (F) satisfy the inequalities (2.1), verify the system of N equations

$$A^{\lambda\mu}\frac{\partial^2 U_S}{\partial x^\lambda \partial x^\mu} + B_S^{T\lambda}\frac{\partial U_T}{\partial x^\alpha} + F_S = 0$$

where U_S denotes any whatsoever of the fifth-order partial derivatives of W_s and where $A^{\lambda\mu}, B_S^{T\lambda}$ and F_S are functions of the variables x^α, of the functions W_s and of their partial derivatives up to the orders one, two and five, respectively.

We have proved that, under the assumptions of Sec. 4, every solution seven times differentiable of the Cauchy problem (with Cauchy data φ_s, ψ_s) takes, as well as its partial derivatives up to the sixth order, some given values continuous and bounded in the considered domain of the initial surface.

We apply to the equations (2.3) the results of chapter I and we add to the integral equations obtained the integral equations (3.1).

Let us sum up the assumptions made and the results obtained.

Assumptions

(A) In the domain D defined by $|x^i - \overline{x}^i| \le d,\ |x^4| \le \varepsilon$ and for values of the unknown functions satisfying

$$|W_s - \overline{\varphi}_s| \le l,\ \left|\frac{\partial W_s}{\partial x^4} - \overline{\psi}_s\right| \le l,\ \left|\frac{\partial W_s}{\partial x^i} - \overline{\frac{\partial \varphi_s}{\partial x^i}}\right| \le l:$$

(1^o) The coefficients $A^{\lambda\mu}$ and f_s have partial derivatives with respect to all their arguments up to the fifth order continuous and bounded, the derivatives of order five satisfying some Lipschitz conditions;

(2^o) The quadratic form $A^{\lambda\mu}X_\lambda X_\mu$ is of normal hyperbolic form: $A^{44} > 0$ and the form $A^{ij}X_iX_j$ negative definite.

(B) In the domain of the initial surface $x^4 = 0$, defined by $|x^i - \overline{x}^i| \le d$, the Cauchy data φ_s and ψ_s admit of partial derivatives continuous and bounded up to the orders six and five.

Conclusion. If we consider a solution W_s seven times differentiable of the assigned Cauchy problem, possessing partial derivatives with respect to the x^α up to the sixth order, continuous and bounded and satisfying the inequalities (2.1) in D, it satisfies in this domain the equations F'. The equations F', viewed as linear equations in the unknown functions U_S, satisfy the assumptions of chapter I, and therefore:

There exists a domain $D_0 \subset D$ in which the functions W_s verify the following system of integral equations.

System of integral equations (I)

This system consists of

(1^o) equations having in the first member a function X of the three parameters

$$x^4, \lambda_2, \lambda_3$$

(representatives of a point of the characteristic conoid of vertex $M_0(x_0)$) and of the four coordinates x_0^α of a point $M_0 \in D_0$. These functions X are the functions

$$x^i, p^i, y_i^j, z_i^j, y_{ih}^j, z_{ih}^j, y_{ihk}^j, z_{ihk}^j$$

of chapter I. These equations are of the form

$$X = \int_{x_0^4}^{x^4} E(X) dx^4 + X_0,$$

where X_0, value of X for $x^4 = x_0^4$, is a given function of $x_0^\alpha, \lambda_2, \lambda_3$;

(2^o) equations having in the first member a function

$$\Omega(x_0^\alpha, x_0^4, \lambda_2, \lambda_3)$$

(functions $\omega_s^r, \omega_{si}^r, \omega_{sij}^r$ of chapter I), of the form

$$\Omega = \int_{x_0^4}^{x^4} F(X, \Omega) dx^4 + \Omega_0,$$

where Ω_0, value of Ω for $x^4 = x_0^4$, is a given function of $x_0^\alpha, \lambda_2, \lambda_3$;

(3^0) equations having in the first member a function W of the four coordinates x^α of a point $M \in D$. The functions W are the functions

$$W_s, W_{s\alpha}, W_{s\alpha\beta}, W_{s\alpha\beta\gamma}, W_{s\alpha\beta\gamma\delta}.$$

The equations are of the form

$$W = \int_0^{x^4} G(W, U) dx^4 + W_0,$$

where W_0, value of W for $x^4 = 0$, is a given function of the three variables x^i.

(4^o) equations having in the first member a function U of the four coordinates x_0^α of a point $M_0 \in D_0$. The functions U are the functions U_S, fifth derivatives of W_s. These equations (Kirchhoff formulas) are of the form

$$U = \int_{x_0^4}^0 \int_0^{2\pi} \int_0^\pi H \, dx^4 \, d\lambda_2 \, d\lambda_3 + \int_0^{2\pi} \int_0^\pi I \, d\lambda_2 \, d\lambda_3.$$

The quantities E, F, G, H, I are formally identical to the corresponding quantities evaluated in chapter I for the equations(E) (upon considering the differentiations with respect to the x^α as having been performed). The quantity G is a function W or U. All these quantities are therefore expressed by means of the functions X, Ω, W and U, occurring in the first members of the integral equations considered, and involve the partial derivatives of the $A^{\lambda\mu}$ and f_s with respect to all their arguments, up to the fifth order, and the partial derivatives of the Cauchy data φ_s and ψ_s up to the orders six and five (in the quantity I and by means of W_0).

Solution of the Cauchy problem

In order to solve the Cauchy problem for the nonlinear equations F we might try to solve, independently of these equations, the system of integral equations verified by the solutions (and to prove afterwards that this solution is indeed a solution of the assigned Cauchy problem). Unfortunately, some difficulties arise for this solution: we have shown in chapter I that the quantities occurring under the integral sign (in particular H) are continuous and bounded, upon assuming differentiability of the coefficients $A^{\lambda\mu}$, viewed as given functions of the variables x^α, these conditions not being realized when the functions $W_s, W_{s\alpha}, ..., U_S$ are independent, the quantity $[A^{ij}]\frac{\partial^\sigma}{\partial x^i \partial x^j}\triangle$ will then fail to be bounded and continuous.

In order to solve the Cauchy problem we shall then pass through the intermediate stage of approximate equations F_1, where the coefficients $A^{\lambda\mu}$ will be some given functions of the x^α, obtained by replacing W_s with a given function $W_s^{(1)}$. The quantities occurring under the integration signs of the integral equations verified by the solutions will then be continuous and bounded if the same holds for the functions $W_s ... U_S$ considered as independent. We will then be in a position to solve the integral equations and show that their solution $W_s ... U_S$ is solution of the equations F_1, and that $W_{s\alpha} ... U_S$ are the partial derivatives of W_s; but we need for that purpose, in the general case, to take as function $W_s^{(1)}$ a function six times differentiable (because the integral equations involve fifth derivatives of the $A^{\lambda\mu}$); the obtained solution W_s being merely five times differentiable, it will be impossible for us to iterate the procedure. The method described will be therefore applicable only if the $A^{\lambda\mu}$ depend uniquely on the W_s and not on the $W_{s\alpha}$: it will then be enough to assume the approximation function five times differentiable.

We shall describe in detail the solution of the Cauchy problem in this case in chapter III, and we will apply it to the equations of relativity in Chapter IV.

In the general case, where $A^{\lambda\mu}$ is function of W_s and $W_{s\alpha}$, one can solve the Cauchy problem by passing through the intermediate step of approximate equations, not of the equations (F) themselves, but of equations previously differentiated with respect to the x^α and viewed as integro-differentiaL equations in the unknown functions $W_{s\alpha}$.

CHAPTER III

1 Solution of the Cauchy problem for the case in which the coefficients $A^{\lambda\mu}$ do not depend on first partial derivatives of the unknown functions

We consider in this chapter a system (F) of n partial differential equations of second order with n unknown functions and four variables, of the kind previously studied:

$$A^{\lambda\mu}\frac{\partial^2 W_s}{\partial x^\lambda \partial x^\mu} + f_s = 0, \qquad (G)$$

where the coefficients $A^{\lambda\mu}$ depend only on the variables x^α and the unknown functions W_t, and not on the first partial derivatives $\frac{\partial W_t}{\partial x^\alpha}$ of these functions.

The coefficients f_s are functions, as previously, of the variables x^α, of the unknown functions W_t and of their first partial derivatives $\frac{\partial W_t}{\partial x^\alpha}$.

Formation of a system of integral equations verified from the solutions of equations (G)

We shall obtain a system of integral equations verified by the solutions of equations (G) by applying the methods used in the previous chapter for the equations of general type (F). Let us point out however that, in the case of equations (G), the coefficients $A^{\lambda\mu}$ not containing the first partial derivatives $\frac{\partial W_t}{\partial x^\alpha} = W_{t\alpha}$, it will be enough to apply the results of chapter I to the equations deduced from equations (G) by *four* differentiations with respect to the variables x^α in order to obtain a system of integral equations whose second members do not contain functions other than those occurring in the first members. The calculations performed in Sec. 2, chapter II prove indeed that these equations read as, by denoting with U_S any whatsoever of the fourth derivatives of the unknown functions W_s

$$A^{\lambda\mu}\frac{\partial^2 U_S}{\partial x^\lambda \partial x^\mu} + B_S^{T\lambda}\frac{\partial U_T}{\partial x^\lambda} + F_s = 0.$$

$A^{\lambda\mu}$ depends only on the variables x^α and the functions W_s.

$B_S^{T\lambda}$, which is a sum of first partial derivatives of the functions $A^{\lambda\mu}$, viewed as functions of the variables x^α and of first partial derivatives of a function f_s with respect to the first partial derivatives $W_{r\alpha}$ of the unknown functions, depends on nothing else but the variables x^α, the unknown functions W_r and their first partial derivatives $W_{r\alpha}$.

F_S is a polynomial of the coefficients $A^{\lambda\mu}$ and f_s and of their partial derivatives with respect to all their arguments up to the fourth order, as well as of the functions W_s and of their partial derivatives with respect to the variables x^α up to the fourth order.

The integral equations (J), verified by the bounded solutions and with bounded first derivatives of equations (G'), deduced as in chapter II from the results of chapter I, only involve the coefficients $A^{\lambda\mu}$ and $B_S^{T\lambda}$ and their partial derivatives up to the orders four and two, respectively, as well as the coefficients F_S. One therefore verifies that these equations (J) contain nothing else but partial derivatives of the functions W_s of order higher than four.

We would face clearly, in order to solve the system of integral equations (J) directly, the same difficulty as in the general case: the quantity H under the sign $\int\int\int$ is not bounded in general if $W_s, W_{s\alpha}...U_S$ are independent functions. We shall be able however, in the case in which the $A^{\lambda\mu}$ depend only on the first derivatives of the W_s, to solve the Cauchy problem by using the results obtained on the system of integral equations verified in a certain domain, from the solutions of the given equations (G) in a way that we are going to describe in what follows.

2 Plan of chapter III (Solution of the Cauchy problem)

A. We shall consider a system G_1, approximate version of G, obtained by replacing in $A^{\lambda\mu}$ (and not in f_s) the unknown W_s with some approximate values $\overset{(1)}{W}_s$, satisfying suitable assumptions.

I. We will prove that the system of integral equations J_1, verified by the solutions of the Cauchy problem assigned with respect to the equations G_1, admits of a unique, continuous and bounded solution in a domain D independent of $\overset{(1)}{W}_s$ if one regards it as a system of integral equations with independent unknown functions X, Ω, W, U.

II. We will prove afterwards that the solutions of J_1 that we have found are solutions of the Cauchy problem given for the eqyations G_1 in the whole domain D, and that the functions W_s obtained admit of partial derivatives up to the fourth order equal to $W_{s\alpha}...U_S$ and satisfy the same assumption as $\overset{(1)}{W}_s$. We denote these functions by $\overset{(2)}{W}_s$.

B. The solution of the Cauchy problem for the equations G_1 defines, in light of previous results, a representation of the space of functions $\overset{(1)}{W}_s$ into itself. We prove that this representation admits a fix point, belonging to the space. The correponding functions W_s are solutions of the given equations (G). This solution, unique, possesses partial derivatives continuous and bounded up to the fourth order.

3 Assumptions made in chapter III

(1^o) In the domain D defined by

$$|x^i - \overline{x}^i| \le d, \; |x^4| \le \varepsilon$$

and for values of the unknown functions satisfying

$$|W_s - \overline{\varphi}_s| \le l, \; \left|\frac{\partial W_s}{\partial x^i} - \overline{\frac{\partial \varphi_s}{\partial x^i}}\right| \le l, \; \left|\frac{\partial W_s}{\partial x^4} - \overline{\psi}_s\right| \le l : \tag{3.1}$$

(a) The coefficients $A^{\lambda\mu}$ and f_s admit partial derivatives with respect to all their arguments up to the fourth order, continuous, bounded and satisfying Lipschitz conditions.

(b) The quadratic form $A^{\lambda\mu}X_\lambda X_\mu$ is of normal hyperbolic type, i.e. $A^{44} > 0$, $A^{ij}X_iX_j$ negative definite.

(2^o) In the domain (d) of the initial surface, defined by $|x^i - \overline{x}^i| \leq d$, the Cauchy data φ_s and ψ_s possess partial derivatives continuous and bounded up to the orders five and four, respectively, satisfying some Lipschitz conditions.

4 Approximate equations G_1

We consider a system of equations approximating the system (G), obtained by replacing in the coefficients $A^{\lambda\mu}$ (and not in f_s) the unknown functions with given functions $\overset{(1)}{W}_s$ which admit of partial derivatives continuous and bounded up to the fourth order (we denote them by $\overset{(1)}{W}_{s\alpha}, ..., \overset{(1)}{U}_S$) in the domain D:

$$|x^i - \overline{x}^i| \leq d, \ |x^4| \leq \varepsilon$$

and satisfy the inequalities

$$\left|\overset{(1)}{W}_s - \overline{\varphi}_s\right| \leq l, \ \left|\frac{\partial \overset{(1)}{W}_s}{\partial x^i} - \overline{\frac{\partial \varphi_s}{\partial x^i}}\right| \leq l, \ \left|\frac{\partial \overset{(1)}{W}_s}{\partial x^4} - \overline{\psi}_s\right| \leq l.$$

We write the system obtained:

$$\overset{(1)}{A}{}^{\lambda\mu}\frac{\partial^2 W_s}{\partial x^\lambda \partial x^\mu} + f_s = 0. \qquad (G_1)$$

A solution W_1, six times differentiable and satisfying the inequalities (3.1), of the equations (G_1) verifies therefore, in D, the following equations:

$$\overset{(1)}{A}{}^{\lambda\mu}\frac{\partial^2 U_S}{\partial x^\lambda \partial x^\mu} + \overset{(1)}{B}{}^{T\lambda}_{S}\frac{\partial U_T}{\partial x^\lambda} + \overset{(1)}{F}_S = 0. \qquad (G_1')$$

One sees easily, by virtue of formulas analogous to the formulas of chapter II, that

(1^o) $\overset{(1)}{A}{}^{\lambda\mu}$ is a function of the variables x^α and of the unknown functions $\overset{(1)}{W}_s$;

(2^o) $\overset{(1)}{B}{}^{T\lambda}_{S}$ is a sum of the following functions:

(a) first partial derivatives of the $\overset{(1)}{A}{}^{\lambda\mu}$ viewed as functions of the variables x^α (hence as functions of the variables x^α and of the functions $\overset{(1)}{W}_s$ and $\overset{(1)}{W}_{s\alpha}$);

(b) first partial derivatives of a function f_s with respect to the functions $W_{r\nu}$ (hence of the functions of x^α, W_s and $W_{s\alpha}$).

(3^o) $\overset{(1)}{F}_S$ is a polynomial of the coefficients $\overset{(1)}{A}{}^{\lambda\mu}$ and f_s and of their partial derivatives with respect to all their arguments up to the fourth order, as the functions $\overset{(1)}{W}_s$ and $\overset{(1)}{W}_{s\alpha}$ and of their partial derivatives with respect to the x^α up to the fourth order.

5 Application of the results of chapter I

The coefficients of equations (G_1'), viewed as linear equation of type (E) in the unknown functions U_S, satisfy in the domain D the assumptions of chapter I. There exists therefore a domain $D_0 \subset D$ in which the fifth derivatives U_S of a solution W_s of the given Cauchy problem, which possess partial derivatives continuous and bounded up to the sixth order and satisfy the inequalities (3.1), verify some Kirchhoff formulas, whose first members are the values at the point $M_0 \in D_0$ of these functions U_S. These equations, together with the integral equations having in the first member some auxiliary functions X and Ω, and with some integral equations analogous to (3.1) of chapter II, form a system of integral equations that we denote by J_1.

6 System of integral equations J_1

Let us consider (independently of the initial equations G_1) the set of integral relations J_1 as a system of integral equations with four groups of unknown functions X, Ω, W and U. The system consists of the following four groups of equations:

(1^o) Some integral equations having in the first member a function X of the four coordinates x_0^α and of three parameters $x^4, \lambda_2, \lambda_3$ (functions corresponding to the functions $x^i, p_i, y_i^j, ...z_{ihk}^j$ which define the characteristic conoids). These equations are of the form

$$X(x_0^\alpha; x^4, \lambda_2, \lambda_3) = \int_{x_0^4}^{x^4} E\, dx^4 + X_0(x_0^\alpha, x_0^4, \lambda_2, \lambda_3). \tag{1}$$

X_0 is a given function. (For x^i, p_i, y_i^j... the values of X_0 are $x_0^i, p_i^0, 0$... respectively).

E is a rational function with denominator

$$T^{*4} = \overset{(1)}{A}{}^{*44} + \overset{(1)}{A}{}^{*i4} p_i$$

of the following quantities:

(a) coefficients $\overset{(1)}{A}{}^{\lambda\mu}$ and their partial derivatives with respect to all their arguments up to the fourth order (functions of $\overset{(1)}{W}_s(x^\alpha)$ and x^α where x^i is replaced by the corresponding X function), function $\overset{(1)}{W}_s$ and partial derivatives up to the fourth order;

(b) functions X;

(c) quantities $\overset{(1)}{a}{}^0_{\alpha\beta}$ and $\overset{(1)}{a}{}^{\alpha\beta}_0$, algebraic functions of the values of the coefficients $\overset{(1)}{A}{}^{\lambda\mu}$ for the values x_0^α and $\overset{(1)}{W}_s(x_0^\alpha)$ of their arguments.

(2^o) Equations having in the first member a function Ω of the x_0^α and of the parameters $x^4, \lambda_2, \lambda_3$ (functions corresponding to $\omega_s^r, \omega_{si}^r, \omega_{sij}^r$). These equations are of the form

$$\Omega = \int_{x_0^4}^{x^4} F\, dx^4 + \Omega_0, \tag{2}$$

where Ω_0 is a given function (for $\omega_s^r, \omega_{si}^r, \omega_{sij}^r$ the values of Ω_0 are $\delta_s^r, 0, 0$, respectively).

F is a rational fraction (with denominator $T^{*4} = \overset{(1)}{A}{}^{*44} + \overset{(1)}{A}{}^{*i4} p_i$) of the following quantities:

(a) coefficients $\overset{(1)}{A}{}^{\lambda\mu}$ and $\overset{(1)}{B}{}^{T\lambda}_S$ and partial derivatives with respect to all their arguments up to the orders three and two, respectively (i.e. coefficients $\overset{(1)}{A}{}^{\lambda\mu}, f_s$, and their partial derivatives up to the third order);

(b) functions $\overset{(1)}{W}_s(x^\alpha)$ and their partial derivatives up to the third order and functions

$$W_{s\alpha}(x^\alpha), W_{s\alpha\beta}(x^\alpha), W_{s\alpha\beta\gamma}(x^\alpha)$$

(functions $W(x^\alpha)$). The x^i are always replaced by the corresponding functions X;

(c) functions X and Ω;

(d) quantities $\overset{(1)}{a}{}^0_{\alpha\beta}$ and $\overset{(1)}{a}{}^{\alpha\beta}_0$.

(3^o) Equations having in the first member a function W of the four coordinates x^α. These equations are of the form

$$W(x^\alpha) = \int_0^{x^4} G\, dx^4 + W_0(x^i). \tag{3}$$

W_0 denotes a given function. (For the functions W_s, W_{si}... the values of W_0 are $\varphi_s, \varphi_{si}, ...$), respectively).

G is a function W or a function U.

(4^o) Some Kirchhoff formulas, having in the first member a function U of the four coordinates x_0^α

$$4\pi U(x_0^\alpha) = \int_{x_0^4}^{0} \int_0^{2\pi} \int_0^{\pi} H\, dx^4\, d\lambda_2\, d\lambda_3 + \int_0^{2\pi} \int_0^{\pi} I\, d\lambda_2\, d\lambda_3. \tag{4}$$

(a) H is the product of the square root of a rational fraction with denominator D^* (polynomial of $\overset{(1)}{A}{}^{\lambda\mu}, X, \tilde{X}$ and p_i^0) and numerator 1, with the sum of the two following rational fractions:

(1) A rational fraction H_a with denominator $(D^*)^3(x_0^4 - x^4)T^{*4}$ (which results only from those terms of the operator L_s^r which contain the second partial derivatives of the function σ) whose numerator is a polynomial of the following functions:

$\overset{(1)}{A}{}^{\lambda\mu}$ and their first and second partial derivatives with respect to all their arguments (functions of $\overset{(1)}{W}{}_s(x^\alpha)$ and x^α where x^i is replaced by the corresponding X function).

$\overset{(1)}{W}{}_s(x^\alpha),\ \overset{(1)}{W}{}_{s\alpha}(x^\alpha),\ \overset{(1)}{W}{}_{s\alpha\beta}(x^\alpha)$.

X and $\tilde{X}$. (One has denoted by $\tilde{X}$ the quotient by $x_0^4 - x^4$ of the functions X for which $X_0 = 0$).

$U(x^\alpha)$ and Ω, which only occur in the product $[U_r]\omega_s^r$ in the polynomial considered.

We remark that this polynomial, function of the seven arguments

$$x_0^\alpha, x^4, \lambda_2, \lambda_3,$$

vanishes for $x^4 = x_0^4$.

(2) A rational fraction H_{1b} with denominator $(D^*)^2 T^{*4}$ of the following quantities:

coefficients $\overset{(1)}{A}{}^{\lambda\mu}$, $\overset{(1)}{B}{}_S^{T\lambda}$ and $\overset{(1)}{F}{}_S$, and their partial derivatives of the first two up to the orders two and one, respectively, with respect to the x^α. In other words, coefficients $\overset{(1)}{A}{}^{\lambda\mu}$ and f_s and their partial derivatives with respect to all their arguments up to the fourth order, and functions

$$\overset{(1)}{W}{}_s(x^\alpha) ... \overset{(1)}{U}{}_s(x^\alpha),\ W_s(x^\alpha) ... U_s(x^\alpha);$$

functions X and $\tilde{X}$;

functions Ω and $\tilde{\Omega}$ (one has denoted by $\tilde{\Omega}$ the quotient by $x_0^4 - x^4$ of the functions Ω for which $\Omega_0 = 0$);

(b) I is the value for $x^4 = 0$ of the product of the square root of a rational fraction with denominator D^*, and numerator 1, with a rational fraction having denominator $(D^*)^2\ \overset{(1)}{A}{}^{*44}T^{*4}$ of the following functions:

$\overset{(1)}{A}{}^{\lambda\mu}$ and their first partial derivatives with respect to all their arguments;

first partial derivatives of f_s with respect to $W_{r\nu}$ (they contribute through $\overset{(1)}{B}{}_S^{T\lambda}$), functions of $W_s(x^\alpha), W_{s\alpha}(x^\alpha)$ and X^α;

$\overset{(1)}{W}{}_s(x^\alpha)$ and $\overset{(1)}{W}{}_{s\alpha}(x^\alpha)$;

X and $\tilde{X}$, Ω and $\tilde{\Omega}$;

Cauchy data $\varphi_s(x^i)$ and $\psi_s(x^i)$ and their partial derivatives with respect to the x^i up to the orders five and four, respectively.

7 I. Solution of the system of integral equations J_1

We remark that the system of integral equations J_1 is divided into two groups; on the one hand

$$X = \int_{x_0^4}^{x^4} E\, dx^4 + X_0, \tag{1}$$

on the other hand

$$\Omega = \int_{x_0^4}^{x^4} F\, dx^4 + \Omega_0, \tag{2}$$

$$W = \int_{x_0^4}^{x^4} G\, dx^4 + W_0, \tag{3}$$

$$4\pi U = \int_{x_0^4}^{0} \int_0^{2\pi} \int_0^{\pi} H\, dx^4\, d\lambda_2\, d\lambda_3 + \int_0^{2\pi} \int_0^{\pi} I\, d\lambda_2\, d\lambda_3. \tag{4}$$

The equations (1) do not contain other unknown functions besides the functions X. We shall solve them first.

We remark on the other hand that the function H_a is a known function when the X are known. We shall then be in a position to restrict the quantity H without making assumptions on the derivatives of the functions U and W, viewed as independent, and solve the remaining equations (2), (3) and (4).

We are therefore going to prove that the system of integral equations J_0 *admits a unique solution*, by making use of the assumptions made on the coefficients $A^{\lambda\mu}$ and f_s and of the assumptions on the functions $\overset{(1)}{W}_s$. We shall collect these assumptions under the name of assumptions B and B' and we will state them in the two following paragraphs.

8 Assumptions (B)

(1) In the domain (D) defined by

$$|x^i - \overline{x}^i| \leq d,\ |x^4| \leq \varepsilon$$

and for values of the functions W_s and $W_{s\alpha}$ satisfying:

$$|W_s - \varphi_s| \leq l,\ |W_{si} - \varphi_{si}| \leq l,\ |W_{s4} - \psi_s| \leq l : \tag{8.1}$$

(a) The coefficients $A^{\lambda\mu}$ and f_s admit partial derivatives with respect to all their arguments up to the fourth order, continuous and bounded by a given number.

(b) The quadratic form $A^{\lambda\mu}X_\lambda X_\mu$ is of normal hyperbolic type. The coefficient A^{44} is bigger than a given positive number.

The coefficients $a_0^{\alpha\beta}$ and $a_{\alpha\beta}^0$ relative to the values of the coefficients $A^{\lambda\mu}$ at a point of the previous domain are bounded by a given number.

(2^o) The approximating functions $\overset{(1)}{W}_s$ admit in the domain (D) of partial derivatives up to the fourth order continuous, bounded and satisfying the inequalities

$$\left|\overset{(1)}{W}_s - \varphi_s\right| \leq l,\ \left|\overset{(1)}{W}_{si} - \varphi_{si}\right| \leq l,\ \left|\overset{(1)}{W}_{s4} - \psi_s\right| \leq l$$

and the analogous identities $\left|\overset{(1)}{W} - W_0\right| \leq l$ up to $\left|\overset{(1)}{U}_S - \Phi_S\right| \leq l$.

(3^o) In the domain (d), defined by

$$|x^i - \overline{x}^i| \leq d,$$

the Cauchy data $\varphi_s(x^i)$ and $\psi_s(x^i)$ possess partial derivatives continuous and bounded with respect to the variables x^i up to the orders five and four, respectively.

We will denote by bounds (B) the different bounds occurring in these assumptions (d, ε, l bounds of coefficients and Cauchy data).

9 Assumptions B'

(1^o) In the domain (D) and for values of the functions W_s and $W_{s\alpha}$ satisfying the inequalities (8.1), the partial derivatives of order four of the coefficients $A^{\lambda\mu}$ and f_s satisfy a Lipschitz condition assigned with respect to all their arguments.

(2^o) It then turns out from the assumptions (B) that, in the domain D and for values of the functions W_s satisfying (8.1), the coefficients $a_0^{\alpha\beta}, a_{\alpha\beta}^0$ and their partial derivatives up to the fourth order verify a Lipschitz condition given with respect to their arguments $x_0^\alpha, W_s(x_0^\alpha)$.

(3^o) The partial derivatives of order four of the functions W_s satisfy a Lipschitz condition with respect *to the three arguments* x^i.

From the assumptions (B) 3^o) it results the inequality

$$\left|\overset{(1)}{W}_s(x'^\alpha) - \overset{(1)}{W}_s(x^\alpha)\right| \leq l' \sum |x'^\alpha - x^\alpha|$$

and the analogous inequalities for the partial derivatives of the $\overset{(1)}{W}_s$ up to the third order.

We shall have in addition:

$$\left|\overset{(1)}{U}_S(x'^i, x^4) - \overset{(1)}{U}_S(x^i, x^4)\right| \leq l \sum \left|x'^i - x^i\right|.$$

(4^o) in the domain (d) the partial derivatives of Cauchy data φ_s and ψ_s of orders five and four, respectively, satisfy a Lipschitz condition with respect to the variables x^i.

From the assumptions (B) there resulted the inequality

$$\left|\varphi_s(x'^i) - \varphi_s(x^i)\right| \leq l'_0 \sum \left|x'^i - x^i\right|$$

and the analogous inequalities for the functions ψ_s and the partial derivatives of ψ_s and φ_s up to the orders three and four.

We have in addition:

$$\left|\phi_{sj}(x'^i) - \phi_{sj}(x^i)\right| \leq l' \sum \left|x'^i - x^i\right|,$$

$$\left|\psi_s(x'^i) - \psi_s(x^i)\right| \leq l'_0 \sum \left|x'^i - x^i\right|,$$

where l' and l'_0 are given numbers which satisfy

$$l' > l'_0.$$

We will refer to the bounds occurring in these assumptions as the bounds (B').

10 Solution of equations (1)

We shall solve first the equations (1) defining the characteristic conoid. These nonlinear integral equations, having in the first member a function X, do not contain other unknown functions besides the functions X.

$$X = \int_{x_0^4}^{x^4} E(X) dx^4 + X_0. \tag{1}$$

Functional space Υ

We shall solve the equations (1) by considering a functional space Υ, the m coordinates of a point of Υ (m is the number of functions X) being some functions X_1 continuous and bounded of the seven arguments $x_0^\alpha, x^4, \lambda_2, \lambda_3$ in the domain Λ defined by

$$\left|x_0^i - \overline{x}^i\right| \leq d,\ \left|x_0^4\right| \leq \Upsilon(x_0^i),$$

$$0 \leq x^4 \leq x_0^4,\ 0 \leq \lambda_2 \leq \pi,\ 0 \leq \lambda_3 \leq 2\pi,$$

with $\Upsilon(x_0^i) \leq \varepsilon$ (υ occurring in the assumptions (B)).

The functions X_1 take for $x^4 = x_0^4$ the assigned values X_0. We denote by $\overline{\mathcal{M}}_0$ the point of Υ having coordinates $\overline{X}_0$ (values of the functions X_0 for $x_0^i = \overline{x}^i, x_0^4 = 0$[6]) and we assume that the functions X_1 satisfy the inequalities

$$|X_1 - \overline{X}_0| \leq d \text{ and } |X_1 - X_0) \leq M\left|x_0^4 - x^4\right|, \tag{10.1}$$

where M is a given number that we will specify later on.

11 Distance of two points of Υ

We shall define in the space Υ the distance of two points $\mathcal{M}_1, \mathcal{M}_1'$ by the maximum in the domain Λ of the sum of absolute values of the differences of their coordinates:

$$d(\mathcal{M}_1, \mathcal{M}_1') = \text{Max}_\Lambda \sum |X_1' - X_1|.$$

The norm introduced in such a way endows the space Υ of the topology of uniform convergence, and one checks easily that the space Υ is a normed, complete and compact space.

12 Representation of the space Υ into itself

To the point $\mathcal{M}_1$ of Υ having coordinates X_1 we associate a point $\mathcal{M}_2$ whose coordinates X_2 are defined by

$$X_2 = \int_{x_0^4}^{x^4} E_1\ dx^4 + X_0. \tag{12.1}$$

E_1 denotes the quantity E occurring in the equations (1), where the functions X are replaced by the corresponding coordinates X_1 of $\mathcal{M}_1$.

Let us show that this representation (12.1) is a representation of the space Υ into itself, i.e. the X_2 are continuous and bounded functions of their seven arguments, take for $x^4 = x_0^4$ the values X_0 and satisfy the same inequalities (10.1) fulfilled by the X_1, if $\varepsilon(x_0^i)$, which defines the domain of variation of the argument x_0^4 of X_1 is suitably chosen.

The E_1 are indeed expressed rationally (cf. Sec. 6) by means of the $\overset{(1)}{W}_{s1}\ \overset{(1)}{A}{}_1^{\lambda\mu}$, of their partial derivatives up to the fourth order (x^i is replaced in all its functions by the corresponding X_1 function),

$$X_1,\ \overset{(1)}{a}{}_0^{\alpha\beta},\ \overset{(1)}{a}{}_{\alpha\beta}^0$$

: all these functions are, by virtue of the assumptions (B) and of the assumptions made upon the X_1, functions continuous and bounded of the seven arguments $x_0^\alpha, x^4, \lambda_2, \lambda_3$. On the other hand, the denominator of the functions E_1 is

$$\overset{(1)}{T}{}_1^{*4} = \left(\overset{(1)}{A}{}^{*44} + \overset{(1)}{A}{}^{*i4} p_i\right)_1$$

[6]With the exception of the functions x^i, for which $X_0 = x_0^i$, the functions X_0 are constants or functions of λ_2, λ_3 only.

and takes the value 1 for $x^4 = x_0^4, X_1 = X_0$. It follows immediately from the assumptions (B) and (B') and from the inequalities verified by the X_1 that $\overset{(1)}{T}{}^{*4}$ satisfies some Lipschitz conditions

$$\left|\overset{(1)}{T}{}_1^{*4} - 1\right| \leq T' \left\{\sum |X_1 - X_0| + |x^4 - x_0^4|\right\} \leq T'(m\, M + 1) \left|x_0^4 - x^4\right|,$$

where T' depends only on the bounds (B) and (B').

We shall therefore be in a position to choose $\varepsilon(x_0^i)$ sufficiently small so that the denominator considered differs from zero in Λ. For example, for

$$\varepsilon(x_0^i) \leq \frac{1}{2T'(m\, M + 1)} \tag{12.2}$$

we shall have in the domain Λ

$$\left|\overset{(1)}{T}{}^{*4}\right| \geq \frac{1}{2}. \tag{12.3}$$

The quantities E_1 are then continuous functions of the seven arguments $x_0^\alpha, x^4, \lambda_2, \lambda_3$ in the domain Λ, and are bounded by a number M which depends only on the (B) bounds

$$E_1 \leq M.$$

The functions X_2 are therefore functions continuous and bounded of their seven arguments, They fulfill the inequalities

$$|X_2 - X_0| \leq M \left|x_0^4 - x^4\right|. \tag{12.4}$$

It will be therefore enough to take $\varepsilon(x_0^i)$ in such a way that

$$\varepsilon(x_0^i) \leq \frac{d - |x_0^i - \overline{x}_0^i|}{M},$$

in order to obtain

$$\left|X_2 - \overline{X}_0\right| \leq d.$$

(Let us remark that the number M of the inequality (10.1) has been chosen in such a way that the functions X_2 verify the same inequality as the functions X_1, cf. (12.4)).

The point $\mathcal{M}_2$ will be therefore a point of Υ if $\varepsilon(x_0^i)$ verifies the inequalities (12.2) and (12.5).

13 The representation reduces the distances

Let us show that the distance of two representative points $\mathcal{M}_2, \mathcal{M}_2'$ is less than the distance of the initial points $\mathcal{M}_1, \mathcal{M}_1'$ if $\varepsilon(x_0^i)$ is suitably chosen.

We deduce immediately from the equations (12.1) the inequality

$$|X_2' - X_2| \leq \left|x_0^4 - x^4\right| \cdot \mathrm{Max}|E_1' - E_1|. \tag{13.1}$$

The E_1 being rational fractions with nonvanishing denominators of bounded functions verifying Lipschitz conditions with respect to the X_1 (the X_1 verifying the assumptions (10.1) we can indeed exploit the assumptions B'). We have on the other hand

$$|E_1' - E_1| \leq M' \cdot \sum |X_1' - X_1|,$$

where M' is a number which depends only on the bounds B and B'. From which

$$d(\mathcal{M}_2, \mathcal{M}_2') \leq mM' \cdot \mathrm{Max}_\Lambda \varepsilon(x_0^i) \cdot d(\mathcal{M}_1, \mathcal{M}_1').$$

In order for the representation (12.1) of the space Υ into itself to reduce the distances it will be therefore enough that $\varepsilon(x_0^i)$ satisfies

$$\varepsilon(x_0^i) < \frac{1}{mM'}. \tag{13.2}$$

We shall therefore choose $\varepsilon(x_0^i)$ as satisfying the inequalities (12.2), (12.5) and (13.2). The representation (12.1) of the space Υ normed, complete and compact into itself, reducing the distances, will then admit a unique fixed point belonging to this space.

Conclusion. In the domain

$$\left|x_0^i - \overline{x}^i\right| \leq d,\ |x_0^4| < \varepsilon(x_0^i),\ 0 \leq x^4 \leq x_0^4,\ 0 \leq \lambda_2 \leq \pi,\ 0 \leq \lambda_3 \leq 2\pi \tag{13.3}$$

the system of integral equations (1) admits a unique solution, continuous and bounded, verifying the inequalities

$$|X - \overline{X}_0| \leq d. \tag{13.4}$$

We remark in particular that the *three functions X corresponding to the x^i define, with the variable x^4, a point belonging to the domain D.*

14 Properties of the functions X. Functions $\tilde{X}$

The functions X verify the equations

$$X = X_0 + \int_{x_0^4}^{x^4} E\, dx^4.$$

The quantities E, not involving[7], besides the X, any functions but the $\overset{(1)}{A}{}^{\lambda\mu}$ and their partial derivatives (given functions of the x^α), possessing the same properties discussed in chapter I. Proofs identical to those performed in chapter I[8] (Sec. 15) show therefore that:

(1^o) The functions $\frac{X-X_0}{x_0^4-x^4}$ are continuous and bounded in Λ. The functions $\tilde{X}$, quotients by $x_0^4 - x^4$ of the X which vanish for $x_0^4 = x^4$, are continuous and bounded in Λ:

$$|X - X_0| < M|x_0^4 - x^4|,\ |\tilde{X}| \leq M.$$

(2^o) The functions

$$\frac{\tilde{X} - \tilde{X}_0}{x_0^4 - x^4} = \frac{\int_{x_0^4}^{x^4}(E - E_0)dx^4}{x_0^4 - x^4}$$

(where $\tilde{X}_0, E_0$ denote the values for $x^4 = x_0^4$ of $\tilde{X}, E$) are continuous and bounded in Λ. The bound on these functions is deduced from the Lipschitz conditions, verified by E (rational fraction bounded from bounded functions verifying some Lipschitz conditions) with respect to the X and x^4:

$$|E - E_0| \leq M'' \left\{ \sum |X - X_0| + |x^4 - x_0^4| \right\}.$$

M'' depends only on the bounds B and B'. We have therefore

$$\left|\tilde{X} - \tilde{X}_0\right| \leq \frac{M}{2}(M\, m + 1)|x^4 - x_0^4|. \tag{14.1}$$

[7] The proofs have been performed in chapter I by using the variable λ_1; it is clear that one can repeat it with the variable x^4, the denominator $\overset{(1)}{T}{}^{*4}$ here introduced being a (nonvanishing) polynomial of the same functions on which E depends.

[8] Since the X found satisfy $|X - \overline{X}_0| \leq d$ we can evaluate $\overset{(1)}{A}{}^{\lambda\mu}(x^\alpha)$ by replacing x^i with the corresponding X function.

(3^o) *The functions X verify Lipschitz conditions with respect to the x_0^i.*

It is sufficient, in order to prove it, to impose on the space Υ the following supplementary assumption:

The functions X_1 verify a Lipschitz condition with respect to the x_0^i

$$\left|X_1\left({x'}_0^i, x_0^4, ...\right) - X_1\left(x_0^i, x_0^4, ...\right)\right| \leq d' \sum \left|{x'}_0^i - x_0^i\right|, \tag{14.2}$$

where d' is a given number.

We have

$$X_2\left({x'}_0^i, ...\right) - X_2\left(x_0^i, ...\right) = \int_{x_0^4}^{x^4} \left(E_1\left({x'}_0^i, ...\right) - E_1(x_0^i, ...)\right) dx^4.$$

$E_1({x'}_0^i)$ and $E_1(x_0^i)$ are evaluated with the help of the functions $X_1({x'}_0^i, ...)$ (in particular $x_1^i({x'}_0^i, ...)$) and $X_1(x_0^i, ...)$, respectively. Since the quantity E_1 verifies a Lipschitz condition with respect to the X_1, one deduces from (14.2):

$$\left|X_2({x'}_0^i, ...) - X_2(x_0^i, ...)\right| \leq |x_0^4 - x^4| M' d' \left|{x'}_0^i - x_0^i\right|,$$

from which, for $\varepsilon(x_0^i) \leq \frac{1}{M'}$, one has

$$\left|X_2({x'}_0^i, ...) - X_2(x_0^i, ...)\right| \leq d' \sum \left|{x'}_0^i - x_0^i\right|.$$

The point $\mathcal{M}_2$, representative of $\mathcal{M}_1$ by virtue of (12.1), is still, with the supplementary assumption made, a point of Υ, and the fixed point has coordinates verifying

$$\left|X({x'}_0^i, ...) - X(x_0^i, ...)\right| \leq d' \sum \left|{x'}_0^i - x_0^i\right|$$

and

$$\left|X({x'}_0^i, ...) - X(x_0^i, ...)\right| \leq |x_0^4 - x^4| M' d' \sum \left|{x'}_0^i - x_0^i\right|$$

from which in particular

$$\left|\tilde{X}({x'}_0^i, ...) - \tilde{X}(x_0^i, ...)\right| \leq M' d' \sum \left|{x'}_0^i - x_0^i\right|.$$

15 Solution of equations (2), (3) and (4)

We now consider the system of integral equations with three groups of unknown functions Ω, W and U, obtained by replacing in the equations (2), (3) and (4) the functions X with the solutions found of equations (1):

$$\Omega = \int_{x_0^4}^{x^4} F \, dx^4 + \Omega_0, \tag{2}$$

$$W = \int_0^{x^4} G \, dx^4 + W_0, \tag{3}$$

$$U = \int_{x_0^4}^0 \int_0^\pi \int_0^{2\pi} H \, dx^4 \, d\lambda_2 \, d\lambda_3 + \int_0^\pi \int_0^{2\pi} I \, d\lambda_2 \, d\lambda_3. \tag{4}$$

16 Functional space $\mathcal{F}$

We shall solve these equations by considering a functional space $\mathcal{F}$, the coordinates of a point of $\mathcal{F}$ being defined in the following way:

(1^o) m_1 of these coordinates (m_1 is the number of functions Ω) are functions Ω_1 continuous and bounded of the seven arguments $x_0^\alpha, x^4, \lambda_2, \lambda_3$ in the domain Λ:

$$\left|x_0^i - \overline{x}^i\right| \leq d, \ \left|x_0^4\right| \varepsilon(x_0^i), \ 0 \leq x^4 \leq x_0^4, \ 0 \leq \lambda_2 \leq \pi, \ 0 \leq \lambda_3 \leq 2\pi.$$

These functions take for $x^4 = x_0^4$ the given values Ω_0 and satisfy the inequalities

$$|\Omega_1 - \Omega_0| \leq h, \tag{16.1}$$

where h is a given number.

We shall suppose in addition

$$|\Omega_1 - \Omega_0| \leq N \left|x^4 - x_0^4\right|,$$

where N is a number that we are going to specify later on. The functions $\tilde{\Omega}_1$, quotients by $x^4 - x_0^4$ of the functions Ω_1 that vanish identically for $x^4 = x_0^4$, are then bounded in the domain Λ:

$$|\Omega_1| \leq N. \tag{16.2}$$

The functions Ω_1 will be assumed continuous in Λ.

(2^o) m_2 of these coordinates (m_2 is the number of functions W and U) are functions W_1, U_1 continuous and bounded of the four variables x^α in the domain (D):

$$\left|x^i - \overline{x}^i\right| \leq d, \ |x^4| \leq \varepsilon(x_0^i).$$

These functions take for $x^4 = 0$ the values W_0 and U_0 defined by the Cauchy data and satisfy the inequalities

$$|W_1 - W_0| \leq l, \ |U_1 - U_0| \leq l. \tag{16.3}$$

(l is the same number occurring in the assumptions B). The functions

$$\Omega_0, W_0, U_0$$

define a point $\mathcal{M}_0 \in \mathcal{F}$.

17 Distance of two points of $\mathcal{F}$

We define in the space $\mathcal{F}$ the distance of two points $\mathcal{M}_1$ and $\mathcal{M}_1'$ by the sum of the upper bounds, in the respective variation domains of their arguments, of the absolute values of differences of their coordinates:

$$d(\mathcal{M}_1, \mathcal{M}_1') = \text{Max}\left\{\sum |\Omega_1' - \Omega_1| + \sum |W_1' - W_1| + \sum |U_1' - U_1|\right\}.$$

The space $\mathcal{F}$ is then, like the space Υ, a normed space, complete (topology of uniform convergence) and compact.

18 Representation of the space $\mathcal{F}$

To the point $\mathcal{M}_1$ of the space $\mathcal{F}$ we associate a point $\mathcal{M}_2$ whose coordinates Ω_2, W_2, U_2 are defined by

$$\Omega_2 = \int_{x_0^4}^{x^4} F_1 \, dx^4 + \Omega_0,$$

$$W_2 = \int_0^{x^4} G_1 \, dx^4 + W_0, \tag{18.1}$$

$$4\pi U_2 = \int_{x_0^4}^0 \int_0^{2\pi} \int_0^{\pi} H_1 \, dx^4 + \int_0^{2\pi} \int_0^{\pi} I_1 \, d\lambda_2 \, d\lambda_3.$$

F_1, G_1, H_1, I_1 denote the quantities F, G, H, I, occurring in the equations (2), (3) and (4), evaluated with the help of the functions X, solutions of equations (1), and by replacing the unknown functions Ω, W, U with the coordinates Ω_1, W_1, U_1 of the point $\mathcal{M}_1$.

Let us prove that the representation (18.1) is a representation of the space $\mathcal{F}$ into itself if $\varepsilon(x_0^i)$ is suitably chosen.

(1^o) F_1 is expressed rationally (cf. Sec. 6) by means, on the one hand, of the $\overset{(1)}{A}{}^{\lambda\mu}, f_s, \overset{(1)}{W}_s$, of their partial derivatives up to the third order and of the $\overset{(1)}{a}{}_0^{\alpha\beta}$ and $\overset{(1)}{a}{}^0_{\alpha\beta}$ (given functions of the X), on the other hand of the Ω_1. All these functions are continuous and bounded functions of the seven arguments $x_0^\alpha; x^4, \lambda_2, \lambda_3$. The denominator $\overset{(1)}{T}{}^{*4}$ of these fractions F_1 being nonvanishing ($\overset{(1)}{T}{}^{*4} \geq \frac{1}{2}$ by virtue of (12.3)), the F_1 are continuous and bounded functions of the $x_0^\alpha, x_4, \lambda_2, \lambda_3$:

$$|F_1| \leq N,$$

N depending only on the bounds B and on h.

The Ω_2 and $\tilde{\Omega}_2$ are therefore continuous and bounded functions of their arguments, and verify

$$|\Omega_2 - \Omega_0| \leq N|x_0^4 - x^4|, \ \tilde{\Omega}_2 \leq N. \tag{18.2}$$

If $\varepsilon(x_0^i)$ satisfies $\varepsilon(x_0^i) \leq \frac{h}{N}$ we shall have

$$|\Omega_2 - \Omega_0| \leq h.$$

Ω_2 satisfies then the same conditions as Ω_1, the number N (upper bound of the F_1 in Λ), occurring in the inequality (16.2), having been chosen so that this is true as well.

(2^o) G_1 being an U_1 or a W_1, the W_2 are continuous and bounded in D by a number P depending only on the bounds (B)

$$|W_2 - W_0| \leq |x^4 P|,$$

from which, for $\varepsilon(x_0^i) \leq \frac{l}{p}$,

$$|W_2 - W_0| \leq l.$$

(3^o) Let us show that the functions H_1 are bounded by a number which only depends on the bounds (B), (B') and on h.

(a) Let us consider the quantity $\overset{(1)}{D}{}^*$ occurring in the denominator: $\overset{(1)}{D}{}^*$ is a polynomial of the functions $\overset{(1)}{A}{}^{*\lambda\mu}, X, \tilde{X}$ and p_i^0 which takes the value -1 for $x^4 = x_0^4$ and $X = X_0$. By virtue of the inequalities (14.2) and (13.3), verified by the functions x^i and the variable x^4 in the domain Λ, $\overset{(1)}{A}{}^{\lambda\mu}$ verifies Lipschitz conditions with respect to the x^α in Λ. One obtains therefore some inequalities verified by the functions X and $\tilde{X}$ and some assumptions (B) stating that

$$\left|\overset{(1)}{D}{}^* + 1\right| \leq D' \left\{\sum |X - X_0| + |x^4 - x_0^4|\right\} \leq D'(m\, M + 1)\varepsilon(x_0^i),$$

where D' is a number which depends only on the bounds (B) and (B'). We shall be therefore able to choose $\varepsilon(x_0^i)$ sufficiently small so that $\overset{(1)}{D}{}^*$ does not vanish. We see for example that

$$\varepsilon(x_0^i) \leq \frac{1}{2D'(m\, M + 1)}$$

leads to

$$\left|\overset{(1)}{D}{}^*\right| \geq \frac{1}{2} \text{ in } \Lambda. \tag{18.3}$$

(b) Let us consider the rational fraction H_{1a} (cf. Sec. 6) with denominator

$$\left(\overset{(1)}{D}{}^*\right)^3 (x_0^4 - x^4)\overset{(1)}{T}{}^{*4}.$$

Its numerator is the product by $\left([U_R]_1\omega^R_{s_1}\right)$ of a polynomial p of the functions $\overset{(1)}{A}{}^{\lambda\mu}, \overset{(1)}{W}{}_s$, of their first and second partial derivatives and of the functions $X, \tilde{X}$ and p_i^0: quantities that are all known, possessing the same properties as in chapter I (Sec. 15). The quotient by $x_0^4 - x^4$ of the polynomial p (which vanishes for $x^4 = x_0^4$) is therefore a function continuous and bounded in Λ. The bound of this function is deduced from the Lipschitz conditions verified by p (polynomial of bounded functions verifying some Lipschitz conditions with respect to the X and x^4):

$$p \leq P'\left\{\sum |X - X_0| + |x^4 - x_0^4|\right\}.$$

P' is a number which depends only on the bounds B and B'.

We have therefore

$$\frac{p}{x_0^4 - x^4} \leq P'(m\ M + 1).$$

The H_{1a} can be therefore put in the form of fractions with numerator

$$[U_R]_1\omega^R_{s_1}\frac{p}{x_0^4 - x^4},$$

continuous and bounded in Λ, with denominator $\overset{(1)}{D}{}^*\ \overset{(1)}{T}{}^{*4}$ continuous and bounded in Λ. The H_{1a} are therefore continuous and bounded in Λ, their bound depending only on the bounds B, B' and h.

(c) The H_{1b} (cf. Sec. 6), rational fractions with nonvanishing denominator of the functions continuous and bounded in Λ, are continuous and bounded in Λ. We see eventually that the H_1 are continuous and bounded in Λ:

$$|H_1| \leq Q,$$

where Q depends on nothing else but B, B' and h.

(3^o) Let us consider I_1. Let us recall that

$$I = \left\{E_S^{i\,*}\ \frac{D^* p_i}{T^{*4}}(x_0^4 - x^4)^2 \sin\lambda_2\right\}_{x^4=0}. \tag{18.4}$$

The $E_S^{*\,i}$ being given by the equality of chapter I involve the partial derivatives of the σ^R_S with respect to the x^i of first order only, and linearly; the results of chapter I show then that the $E^{i\,*}_{S_1}(x_0^4 - x^4)^2$ are continuous and bounded in Λ because $X, \tilde{X}, \overset{(1)}{D}, \overset{(1)}{D}{}^*$ and their partial derivatives possess the same properties as in chapter I, and that the Ω_1 and $\tilde{\Omega}_1$ are continuous and bounded. We remark in addition that the products of all terms of the $(E_S^{i\,*})_1$ by $x_0^4 - x^4$ are bounded (cf. chapter I and the previous inequalities) by a number R_1 depending on nothing else but the bounds B, B' and h, with the exception of the term

$$-[U_R]_1\ \omega^R_{S_1}\left[\overset{(1)}{A}{}^{ij}\right]\frac{\partial\overset{(1)}{\sigma}}{\partial x^j}. \tag{18.5}$$

We have therefore

$$I_1 \leq R_1|x_0^4| + \Phi_R\left(\omega_{S_1}^R\right)_{x^4=0}\left\{\overset{(1)}{A}{}^{ij}\frac{\partial\overset{(1)}{\sigma}}{\partial x^j}p_i\frac{\overset{(1)}{D}{}^*}{\overset{(1)}{T}{}^{*4}}(x_0^4 - x^4)^2\right\}_{x^4=0}\sin\lambda_2. \tag{18.6}$$

The quantity enclosed in brackets, J, is a known quantity, which verifies a Lipschitz condition with respect to the functions $X, \tilde{X}$ and the variable x^4 and which takes the value 1 for $x^4 = x_0^4$. We have therefore in Λ:

$$|J - 1| \leq R_2|x^4 - x_0^4| \text{ and } |(J)_{x^4=0} - 1| \leq R_2|x_0^4|, \tag{18.7}$$

where R_2 is a number that only depends on the bounds B, B' and h. We deduce on the other hand from the inequality (16.1), verified by the functions Ω,

$$\left|\left(\omega_{S_1}^R\right)_{x^4=0} - \delta_S^R\right| \leq N|x_0^4|. \tag{18.8}$$

We deduce from the inequalities (18.6), (18.7), (18.8)

$$|I_1 - \Phi_S \sin\lambda_2| \leq R_3|x_0^4|,$$

where R_3 is a number which depends on nothing else but B, B' and h.

The previous inequality is verified at every point $x^i(x_0^i, 0, \lambda_2, \lambda_3)$ of the domain d. We have assumed on the other hand (assumptions B') that the Φ_S were verifying some Lipschitz conditions with respect to the x^i:

$$\left|\Phi_S(x^i) - \Phi_S(x_0^i)\right| \leq l_0'|x^i - x_0^i|.$$

The x^i verify (cf. (13.4)) $|x^i - x_0^i| \leq M_1|x_0^4 - x^4|$ and, having taken here for value $x^4 = 0$, we have

$$\left|\Phi_S(x^i) - \Phi_S(x_0^i)\right| \leq l_0' M|x_0^4|. \tag{18.9}$$

We see eventually that there exists a number R, depending on nothing else but the bounds (B), (B') and h, such that

$$\left|I_1 - \Phi_S(x_0^i)\sin\lambda_2\right| \leq R|x_0^4|.$$

The functions

$$U_2 = \frac{1}{4\pi}\int_{x_0^4}^0\int_0^{2\pi}\int_0^{\pi} H_1\, dx^4\, d\lambda_2\, d\lambda_3 + \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I_1\, d\lambda_2\, d\lambda_3$$

are hence continuous and bounded functions of the x_0^α and verify, $\Phi_S(x_0^i)$ having been denoted by U_0, the inequality

$$|U_2 - U_0| \leq |x_0^4|\frac{\pi}{2}(Q + R),$$

from which, for

$$\varepsilon(x_0^i) \leq \frac{2l}{\pi(Q + R)} \tag{18.10}$$

we shall have

$$|U_2 - U_0| \leq l.$$

The functions Ω_2, W_2, U_2 possess then the same properties as Ω_1, W_1, U_1. The point $\mathcal{M}_2$ is hence a point of $\mathcal{F}$ if (x_0^i) verifies, besides the inequalities that were imposed upon it in the solution of equations (1), the inequalities (18.10), (18.2), (18.9).

19 Distance of two representative points

Let us evaluate the distance of the points $\mathcal{M}_2, \mathcal{M}'_2$ representative of $\mathcal{M}_1, \mathcal{M}'_1$. We shall deduce from the equations (16.3), defining the representation, that in the domain Ω we have

$$\Omega'_2 - \Omega_2 \leq |x^4_0 - x^4| \text{Max}_\Lambda |F'_1 - F_1|. \tag{1^o}$$

It turns out from the expression F_1, from the assumptions (B) and the assumptions made on Ω_1 and W_1 that F_1 verifies a Lipschitz condition with respect to the functions Ω_1 and W_1 whose N' coefficient depends on nothing else but the bounds (B) and h. From which the inequality

$$|\Omega'_2 - \Omega_2| \leq N'|x^4_0 - x^4| \text{Max} \left\{ \sum |\Omega'_1 - \Omega_1| + \sum |W'_1 - W_1| \right\}. \tag{19.1}$$

$$|W'_2 - W_2| \leq |x^4| \text{Max}_D |G'_1 - G_1|. \tag{2^o}$$

G_1 being a function W_1 or a function U_1, we have

$$|W'_2 - W_2| \leq |x^4| \text{Max}_D \left\{ \sum |W'_1 - W_1| + \sum |U'_1 - U_1| \right\}.$$

$$|U'_2 - U_2| \leq \frac{\pi}{2} |x^4_0| \text{Max}_D |H'_1 - H_1| + \frac{\pi}{2} \text{Max}_d (I'_1 - I_1). \tag{3^o}$$

(a) It turns out from the expression of H_1 (in particular from the fact that the polynomial p, occurring in the numerator of the function H_a, in independent of the point $\mathcal{M}_1$ in $\mathcal{F}$ that we consider), from the assumptions (B) and from the inequalities of Sec. 18 that H_1 verifies a Lipschitz condition with respect to the functions $\Omega_1, \tilde{\Omega}_1, W_1, U_1$ whose R'_1 coefficient depends only on the bounds (B), (B') and h:

$$|H'_1 - H_1| \leq R'_1 \Bigg\{ \sum |\Omega'_1 - \Omega| + \sum |\tilde{\Omega}'_1 - \tilde{\Omega}_1| + \sum |W'_1 - W_1| + \sum |U'_1 - U_1| \Bigg\}.$$

(b) Let us consider the quantity I_1, given by the equality (18.1), where the only unknown functions are the functions $(\Omega_1)_{x^4=0}$. The expression of the E^i_S (in particular the one of $\frac{\partial \omega^R_S}{\partial x^i}$), the results of chapter I and those obtained from the solution of equations (1), the assumptions (B) and those made upon Ω_1 show that the product

$$\left\{ E^i_{S_1} (x^4_0 - x^4)^2 \right\}_{x^4=0}$$

verifies a Lipschitz condition with respect to the functions $(\Omega_1)_{x^4=0}$ whose R'_2 coefficient depends only on the bounds (B), (B') and h:

$$|I'_1 - I_1| \leq R'_2 \sum |\Omega'_1 - \Omega_1|_{x^4=0}.$$

We have therefore

$$\begin{aligned} |U'_2 - U_2| \quad &\leq \quad R'_2 |x^4_0| \text{Max}_D \Bigg\{ \sum |\Omega'_1 - \Omega_1| + \sum \left| \tilde{\Omega}'_1 - \Omega_1 \right| \\ &+ \quad \sum |W'_1 - W_1| + \sum |U'_1 - U_1| \Bigg\} \\ &+ \quad \frac{\pi}{2} R'_2 \text{Max}_d \sum |\Omega'_1 - \Omega_1|_{x^4=0}. \end{aligned} \tag{19.2}$$

Let us then consider the point $\mathcal{M}_3$ representative of the point $\mathcal{M}_2$ (i.e. obtained starting from $\mathcal{M}_2$ with the help of equalities analogous to (18.1)). The transformation mapping $\mathcal{M}_1$ into $\mathcal{M}_3$

is a representation of the space $\mathcal{F}$ into itself. Let us compute the distance of two representative points.

We shall deduce from the inequality (19.1)

$$\left|\tilde{\Omega}_2' - \tilde{\Omega}_2\right| \leq N' \mathrm{Max}_\Lambda \left\{\sum |\Omega_1' - \Omega_1| + \sum |W_1' - W_1|\right\}. \tag{19.3}$$

The inequalities (19.1), (19.2) and (19.3), written one after the other for the representations $\mathcal{M}_1 \to \mathcal{M}_2$ and $\mathcal{M}_2 \to \mathcal{M}_3$, show then without any difficulty that there exists a number α, nonvanishing, depending on nothing else but the bounds (B), (B') and h such that, for

$$\varepsilon(x_0^i) < \alpha$$

one has

$$d(\mathcal{M}_3, \mathcal{M}_3') < kd(\mathcal{M}_1, \mathcal{M}_1'),$$

where k is a given number less than 1.

The representation of the space $\mathcal{F}$ into itself which leads from $\mathcal{M}_1$ to $\mathcal{M}_3$ admits then a unique fix point, and the same holds for the representation (18.1) originally given.

20 Conclusion

There exists a number $\varepsilon(x_0^i)$ depending only on the bounds (B), (B') and h (and nonvanishing) such that, in the respective domains:

$$\left|x_0^i - \overline{x}^i\right| \leq d,\ |x_0^4| \leq \varepsilon(x_0^i),\ 0 \leq x^4 \leq x_0^4,\ 0 \leq \lambda_2 \leq \pi,\ 0 \leq \lambda_3 \leq 2\pi \tag{1}$$

$$\left|x^i - \overline{x}^i\right| \leq d,\ |x^4| \leq \varepsilon(x_0^i). \tag{2}$$

The equations (2), (3) and (4) have a unique solution, continuous and bounded $\Omega(x_0^\alpha, x^4, \lambda_2, \lambda_3)$ and $W(x^\alpha), U(x^\alpha)$ verifying the inequalities

$$|\Omega - \Omega_0| \leq h,\ |W - W_0| \leq l,\ |U - U_0| \leq l.$$

We shall prove in addition that the functions W and U obtained satisfy, as $\overset{(1)}{W}$ and $\overset{(1)}{U}$, some Lipschitz conditions with respect to the variables x^i.

21 The functions $W(x^\alpha)$ and $U(x^\alpha)$ fulfill Lipschitz conditions with respect to the variables x^i

In order to prove that the functions W and U, solutions that we have found of equations (2), (3) and (4) satisfy Lipschitz conditions with respect to the x^i it is enough to make on the functional space $\mathcal{F}$ previously considered the following supplementary assumptions:

Assumptions

(1^o) The functions Ω_1 and $\tilde{\Omega}_1$ satisfy Lipschitz conditions with respect to the three arguments x_0^i

$$\left|\Omega_1(x_0^i, x_0^4, x^4, \lambda_2, \lambda_3) - \Omega_1({x'}_0^i, x_0^4, x^4, \lambda_2, \lambda_3)\right| \leq h' \sum \left|{x'}_0^i - x_0^i\right| \tag{21.1}$$

with $h' \leq \left|x_0^4 - x^4\right| N'$; in particular

$$\left|\tilde{\Omega}_1(x_0^i, ...) - \tilde{\Omega}_1({x'}_0^i, ...)\right| \leq N' \sum \left|{x'}_0^i - x_0^i\right|, \tag{21.2}$$

where h' is an arbitrary given number, N' a number that we will specify later on, function of the previous bounds.

(2^o) The functions W_1 and U_1 satisfy Lipschitz conditions with respect to the x^i:

$$\begin{aligned} \left| W_1({x'}^i, x^4) - W_1(x^i, x^4) \right| &\leq l \sum \left| {x'}^i - x^i \right|, \\ \left| U_1({x'}^i, x^4) - U_1(x^i, x^4) \right| &\leq l \sum \left| {x'}^i - x^i \right|. \end{aligned} \tag{21.3}$$

22 Representation of $\mathcal{F}$ into itself

$\mathcal{F}$, endowed with the previous norm, is still a normed, complete and compact space. Let us show that the representative points $\mathcal{M}_2$ of the points $\mathcal{M}_1 \in \mathcal{F}$ are still points of $\mathcal{F}$ if $\varepsilon(x_0^i)$ is suitably chosen.

(1^o)

$$\Omega_2\left({x'}_0^i, ...\right) - \Omega_2\left(x_0^i, ...\right) = \int_{x_0^4}^{x^4} \left(F_1\left({x'}_0^i, ...\right) - F_1\left(x_0^i, ...\right)\right) dx^4. \tag{22.1}$$

The quantities $F_1\left({x'}_0^i, ...\right)$ and $F_1\left(x_0^i, ...\right)$ are evaluated with the help of the functions $X({x'}_0^i, ...)$ (in particular $x^i({x'}_0^i, ...), \Omega_1({x'}_0^i, ...)$

and $x^i(x_0^i, ...), \Omega_1(x_0^i, ...)$),

respectively.

It turns out from the expression of F_1, the assumptions made (in particular from (14.2) and (18.2)) that

$$\begin{aligned} \left| F_1({x'}_0^i, ...) - F_1(x_0^i, ...) \right| &\leq N' \sum \left| {x'}_0^i - x_0^i \right|, \\ \left| \Omega_2({x'}_0^i, ...) - \Omega_2(x_0^i, ...) \right| &\leq \left| x_0^4 - x^4 \right| N' \sum \left| {x'}_0^i - x_0^i \right|, \end{aligned} \tag{22.2}$$

if $\varepsilon(x_0^i)$ satisfies

$$\varepsilon(x_0^i) \leq \frac{h'}{N'}.$$

We shall have therefore

$$\left| \Omega_2({x'}_0^i, ...) - \Omega_2(x_0^i, ...) \right| \leq h' \sum \left| {x'}_0^i - x_0^i \right|. \tag{22.3}$$

If N' denotes the number, that depends only on the bounds (B), (B') and h, occurring in the inequality (21.2), we shall have equally well

$$\left| \tilde{\Omega}_2({x'}_0^i, ...) - \tilde{\Omega}_2(x_0^i, ...) \right| \leq N' \sum \left| {x'}_0^i - x_0^i \right|.$$

(2^o)

$$\begin{aligned} \left| W_2({x'}^i, x^4) - W_2(x^i, x^4) \right| &= \int_0^{x^4} \left(G_1({x'}^i, t) - G_1(x^i, t)\right) dt \\ + \quad W_0({x'}^i) - W_0(x^i). \end{aligned} \tag{22.4}$$

G_1 being a function W_1 or a function U_1, the inequality (21.3) shows, under the assumptions B' on the Cauchy data, that one has

$$\left| W_2({x'}^i, x^4) - W_2(x^i, x^4) \right| \leq |x^4| l \sum \left| {x'}^i - x^i \right| + l_0 \sum \left| {x'}^i - x^i \right|.$$

One then sees that

$$\varepsilon(x_0^i) \leq \frac{l - l_0}{l}$$

implies

$$\left|W_2(x'^i, x^4) - W_2(x^i, x^4)\right| \le l \sum \left|x'^i - x^i\right|.$$

$$U_2(x'^i_0, x^4_0) - U_2(x^i_0, x^4_0) = \int_{x^4_0}^{0} \int_0^{\pi} \int_0^{2\pi} \left[H_1(x'^i_0, ...) - H_1(x^i_0, ...)\right] dx^4 \, d\lambda_2 \, d\lambda_3$$

$$+ \int_0^{2\pi} \int_0^{\pi} \left[I_1(x'^i_0, ...) - I_1(x^i_0, ...)\right] d\lambda_2 \, d\lambda_3. \qquad (3^o))$$

The quantities $H_1(x'^i_0), I_1(x'^i_0)$ and $H_1(x^i_0), I_1(x^i_0)$ are evaluated with the help of the functions $X(x'^i_0, ...)$ (in particular $x^i(x^i_0, ...)$), $\Omega_1(x'^i_0, ...)$ and

$$X(x^i_0, ...), \Omega_1(x^i_0, ...),$$

respectively.

Quantity H_1

(a) *Let us consider the polynomial* p occurring in the denominator of H_{1a}. p is a polynomial of the functions $\left[\overset{(1)}{A}{}^{\lambda\mu}\right], \overset{(1)}{W}_s(x^\alpha)$, of their first and second partial derivatives, of the functions $X, \tilde{X}$ and p^0_i.

The Taylor series expansion of this polynomial, starting from the values

$$\left[\overset{(1)}{A}{}^{\lambda\mu}\right]_0 = \delta^\mu_\lambda, \ \left[\frac{\partial \overset{(1)}{A}{}^{\lambda\mu}}{\partial x^\alpha}\right] = \left[\frac{\partial \overset{(1)}{A}{}^{\lambda\mu}}{\partial x^\alpha}\right]_0, ...,$$

$$\overset{(1)}{W}_s(x^\alpha) = \overset{(1)}{W}_s(x^\alpha_0), \ \overset{(1)}{W}_{s\alpha}(x^\alpha) = \overset{(1)}{W}_{s\alpha}(x^\alpha_0), ..., X = X_0, \tilde{X} = \tilde{X}_0$$

(values of the corresponding functions for the value x^4_0 of the parameter x^4) for which the polynomial p vanishes, shows that p is a polynomial of the functions already listed, and of the functions $\left[\overset{(1)}{A}{}^{\lambda\mu}\right] - \delta^\mu_\lambda, ..., \overset{(1)}{W}_s(x^\alpha) - \overset{(1)}{W}_s(x^\alpha_0), ..., \tilde{X} - \tilde{X}_0, X - X_0$ *whose terms are at least of first degree with respect to the set of these last functions*.

The quantity $\frac{p}{x^4_0 - x^4}$ is therefore a polynomial of the functions

$$\overset{(1)}{A}{}^{\lambda\mu}, ..., \overset{(1)}{W}_s(x^\alpha), ..., X, \tilde{X}, p^0_i$$

and of the functions

$$\frac{\left[\overset{(1)}{A}{}^{*\lambda\mu}\right] - \delta^\mu_\lambda}{x^4_0 - x^4}, \ \frac{\overset{(1)}{W}_s(x^\alpha) - \overset{(1)}{W}_s(x^\alpha_0)}{x^4_0 - x^4}, ..., \frac{X - X_0}{x^4_0 - x^4}, \ \frac{\tilde{X} - \tilde{X}_0}{x^4_0 - x^4}.$$

Since the coefficients $\overset{(1)}{A}{}^{*\lambda\mu}$ and the functions $\overset{(1)}{W}_s$ admit bounded derivatives with respect to the x^α up ro the fourth order, whereas the functions considered involve only derivatives of the first two orders, it turns out from the assumptions (B) and the inequalities (13.4) and (14.1), verified by X and $\tilde{X}$, that all listed functions are bounded in Λ by a number which only depends on the bounds (B) and (B').

The polynomial $\frac{p}{x^4_0 - x^4}$ verifies therefore a Lipschitz condition with respect to each of these functions, whose coefficient depends only on the bounds (B) and (B'). Let us prove that these functions themselves verify Lipschitz conditions with respect to the x^i_0. It will be enough for us, by virtue of the assumptions (B) and the inequalities of Sec. 13 to prove this result for:

(1) the functions $\frac{\left[\overset{(1)}{A}{}^{\lambda\mu}\right]-\delta^\mu_\lambda}{x^4_0-x^4}$ and $\frac{\overset{(1)}{W}_s(x^\alpha)-\overset{(1)}{W}_s(x^\alpha_0)}{x^4_0-x^4}$ and the analogous functions written with first and second partial derivatives of the $\overset{(1)}{A}{}^{*\lambda\mu}$ and $\overset{(1)}{W}_s$ with respect to the x^α;

(2) the functions $\frac{X-X_0}{x^4_0-x^4}$.

(1) Let us set

$$F(x^i_0, x^4_0, x^4, \lambda_2, \lambda_3) = \frac{\overset{(1)}{A}{}^{*\lambda\mu} - \delta^\mu_\lambda}{x^4_0 - x^4},$$

where

$$\overset{(1)}{A}{}^{*\lambda\mu} - \delta^\mu_\lambda = A^{*\lambda\mu}\left(\overset{(1)}{W}_s(x^i, x^4), \overset{(1)}{W}_s(x^i_0, x^4_0), x^i, x^4, x^i_0, x^4_0\right)$$
$$-A^{*\lambda\mu}\left(\overset{(1)}{W}_s(x^i_0, x^4_0), \overset{(1)}{W}_s(x^i_0, x^4_0), x^i_0, x^4_0, x^i_0, x^4_0\right)$$

with $x^i = x^i(x^i_0, x^4_0, x^4, \lambda_2, \lambda_3)$.

Let us consider the quantity $F({x'}^i_0, ...) - F(x^i_0, ...)$. The function occurring in the numerator vanishes for $x^4 = x^4_0$ (because the two functions $F({x'}^i_0, ...)$ and $F(x^i_0, ...)$ vanish) and it admits a derivative with respect to x^4 continuous and bounded in the domain Λ (because the same holds for the functions $F(x^i_0, ...)$ and $\overset{(1)}{A}{}^{*\lambda\mu}, \overset{(1)}{W}_s$ and x^i). We have therefore (formula of finite increments)

$$F({x'}^i_0, ...) - F(x^i_0, ...)$$
$$= \left\{\frac{\partial}{\partial x^4}\left[\left(\overset{(1)}{A}{}^{*\lambda\mu}({x'}^i_0, ...) - \delta^\mu_\lambda\right) - \left(\overset{(1)}{A}{}^{*\lambda\mu}(x^i_0, ...) - \delta^\mu_\lambda\right)\right]\right\}_{x^4 = x^4_0 - \theta(x^4 - x^4_0)},$$

where θ is a number in between 0 and 1.

Since the derivative of the function $\overset{(1)}{A}{}^{*\lambda\mu}({x'}^i_0, ...)$ with respect to the parameter x^4 verifies a Lipschitz condition with respect to the x^i_0 (assumptions B and B', results of Sec. 14), whose coefficient depends only on the bounds (B) and (B'), we see eventually that

$$F({x'}^i_0, ...) - F(x^i_0, ...) \leq L_1 \sum \left|{x'}^i_0 - x^i_0\right|$$

where L_1 depends only on the bounds (B) and (B'). Q.E.D.

The same proof holds for the function $\frac{\overset{(1)}{W}_s(x^\alpha)-\overset{(1)}{W}_s(x^\alpha_0)}{x^4_0-x^4}$ and for the functions built with the partial derivatives of the $\overset{(1)}{A}{}^{*\lambda\mu}$ or $\overset{(1)}{W}_s$ *up to the third order included.*

(2) We have (cf. Sec. 14)

$$\tilde{X} - \tilde{X}_0 = \frac{\int_{x^4_0}^{x^4} (E - E_0) dx^4}{x^4_0 - x^4},$$

from which

$$(\tilde{X} - \tilde{X}_0)_{{x'}^i_0} - (\tilde{X} - \tilde{X}_0)_{x^i_0} = \frac{\int_{x^4_0}^{x^4}\left[(E - E_0)_{{x'}^i_0} - (E - E_0)_{x^i_0}\right] dx^4}{x^4_0 - x^4}.$$

E being a rational fraction with denominator $\overset{(1)}{T}{}^{*4}$ of the coefficients $\overset{(1)}{A}{}^{*\lambda\mu}$ and of their partial derivatives up to the third order (the fourth-order partial derivatives only occur in the equations (1) having in the first member z^i_{jhk}, whereas X correspond only to the functions $y^j_i, y^j_{ih}, y^j_{ihk}$) and of the functions X, we can write $E - E_0$ in the form of rational fraction with denominator $\overset{(1)}{T}{}^{*4}$ (because $\overset{(1)}{T}{}^{*4} = 1$ for $x^4 = x^4_0$) of the previous functions and of the functions $X - X_0$, $\overset{(1)}{A}{}^{*\lambda\mu} - \delta^\mu_\lambda$,

... whose denominator has all its terms of first degree at least with respect to the set of these functions. We can then write

$$E - E_0 = (x_0^4 - x^4)F,$$

where F is a rational fraction with denominator $\overset{(1)}{T}{}^{*4}$ of the previous functions and of the functions

$$\frac{X - X_0}{x_0^4 - x^4},\ \frac{\overset{(1)}{A}{}^{*\lambda\mu} - \delta_\lambda^\mu}{x_0^4 - x^4}, \dots .$$

Since all these functions verify Lipschitz conditions with respect to the x_0^i, it is clear that

$$\left|(E - E_0)_{x'^i_0} - (E - E_0)_{x_0^i}\right| \leq L_2 \left|x_0^4 - x^4\right| \sum \left|x'^i_0 - x_0^i\right|,$$

from which

$$\left|(X - X_0)_{x'^i_0} - (X - X_0)_{x_0^i}\right| \leq \frac{L_1}{2} \left|x_0^4 - x^4\right|$$

and

$$\left|\left(\frac{X - X_0}{x_0^4 - x^4}\right)_{x'^i_0} - \left(\frac{X - X_0}{x_0^4 - x^4}\right)_{x_0^i}\right| \leq \frac{L_2}{2}.\ \text{Q.E.D.}$$

We have thus proven that the quantity $\frac{p}{x_0^4 - x^4}$ verifies a Lipschitz condition, with respect to the x_0^i, whose coefficient depends only on the bounds (B) and (B').

(b) There remains no difficulty to prove that the quantity H_1 (product of the square root of a rational fraction with numerator 1 and nonvanishing denominator with a rational fraction with nonvanishing denominator of the bounded functions verifying all Lipschitz conditions with respect to the x_0^i) verifies in Λ a Lipschitz condition with respect to the x_0^i whose coefficient Q' depends on nothing else but the bounds (B), (B'), h and h'

$$|H_1' - H_1| \leq Q' \sum \left|x'^i_0 - x_0^i\right|.$$

Quantity I_1

One proves easily, by considering the expression of I_1 and the previous inequalities, that all terms of I_1, with the exception of the term (18.4), verify Lipschitz conditions with respect to the x_0^i whose coefficient is of the form $R_1'|x_0^4$, where R_1' is a number depending only on the bounds (B) and (B').

Let us consider the term (18.4). One finds (by a proof analogous to those used for H_1) that $\frac{J(x_0^i) - 1}{x_0^4 - x^4}$ verifies a Lipschitz condition with respect to the variables x_0^i, from which

$$\left|J(x'^i_0) - J(x_0^i)\right|_{x^4=0} \leq R_2'|x_0^4| \sum \left|x'^i_0 - x_0^i\right|,$$

from which, by using the inequality (21.1) and the inequalities of Sec. 18

$$\left|I_0(x'^i_0) - I_1(x_0^i)\right| \leq |x_0^4| R_0'' \sum \left|x'^i_0 - x_0^i\right| + \left|U_0(x'^i_0) - U_0(x_0^i)\right| \left(1 + R_2''|x_0^4|\right).$$

One then obtains Lipschitz conditions, verified by U_0,

$$\left|I_1(x'^i_0) - I_1(x_0^i)\right| \leq \left(R'|x_0^4| + l_0\right) \sum \left|x'^i_0 - x_0^i\right|,$$

where R' is a number depending only on the bounds B and B'.

We shall deduce eventually from the Lipschitz conditions, verified by H_1 and I_1,

$$\left|U_2(x'^i_0, x_0^4) - U_2(x_0^i, x_0^4)\right| \leq \frac{\pi}{2} \left[(Q' + R')|x_0^4| + l_0\right] \sum \left|x'^i_0 - x_0^i\right|$$

hence that the inequality

$$\varepsilon(x_0^i) \leq \frac{l - l_0}{Q' + R'} \frac{2}{\pi}$$

implies

$$\left| U_2({x'}_0^i, x^4) - U_2(x^i, x^4) \right| \leq l \sum |{x'}^i - x^i|.$$

Conclusion. The inequalities of Sec. 22 prove that, if $\varepsilon(x_0^i)$ satisfies the corresponding inequalities, the point $\mathcal{M}_2$ is again, under the additional assumptions made, a point of $\mathcal{F}$. The application of the fixed-point theorem shows that, in the domain D, the functions W and U satisfy Lipschitz conditions with respect to the x^i with coefficient l.

The functions W and U, solutions of the integral equations (J_1), satisfy therefore, in D, the same inequalities holding for the functions $\overset{(1)}{W}, \ldots \overset{(1)}{U}$.

II. Solution of the equations G_1

We will now prove that the functions W_s, solutions of the equations I_1, are solutions of the equations G_1, and that the functions $W_{s\alpha}, \ldots U_S$, solutions of the equations I_1, are the partial derivatives (up to the fourth order) of the W_s, in a domain D depending only on the bounds B and B'. We shall use for the proof the approximation of continuous functions by means of analytic functions (method used in analogous problems by Hadamard and several other authors).

23 Analytic coefficients and analytic Cauchy data

Let us consider some equations G_1 where the coefficients and Cauchy data are analytic ($A^{\lambda\mu}$, f_s,

$$\overset{(1)}{W}_s, \varphi_s$$

and ψ_s analytic functions of their various arguments). The Cauchy problem for the equations G_1 admits an analytic solution in a neighbourhood V of the domain (d) of the surface $x^4 = 0$ carrying the initial data (Cauchy-Kowalevski theorem). If the coefficients and the Cauchy data satisfy the assumptions of chapter II, there exists a neighbourhood V of (d) where this solution satisfies the integral equations I_1.

Let us consider on the other hand, independently of equations G_1, the integral equations I_1. We shall prove in the next section that they admit, within a domain D depending only on the bounds B and B', a unique analytic solution which coincides therefore, in the part shared by the domains V' and D^*, with the solution of equations G_1. This principle of analytic continuation shows then that this solution of equations I_1 is solution of equations G_1 in the whole of D.[9]

24 Analyticity of the solutions of I_1

Let us prove for example the analyticity, in D, of the solution of equations (1)

$$X = \int_{x_0^4}^{x^4} E \, dx^4 + X_0,$$

when E is an analytic function of the quantities X, x_0^α, x^4, by extending its definition to the complex domain:

E being an analytic function of the X, x_0^α, x^4, bounded by M in the domain

$$R(|X - \overline{X}_0| \leq d, \; |x_0^i - \overline{x}^i| \leq d, \; |x^4| \leq |x_0^4| \leq \varepsilon(x_0^i))$$

[9]Being solution of equations G_1 in a domain as close as one wants to D this solution of equations I_1, which is continuous in D, is solution of equations G_1 in D.

of variation of its real arguments, it is expandable in an absolutely convergent series in the neighbourhood of every point of R. We can thus extend the definition of E to a domain of variation of the complex arguments $Z = X + iy,\ z_0^\alpha = x_0^\alpha + iy_0^\alpha,\ z^4 = x^4 + iy^4$ by expressing it in the form of a convergent series, hence holomorphic in the m cylinders V, centred at a point whatsoever of V and defined by

$$|Z' - X| \leq a_X,\ \left|{z'}_0^\alpha - x_0^\alpha\right| \leq b_{x_0^\alpha},\ |z^4 - x^4| \leq C_{x^4}.$$

The partial derivatives $\frac{\partial E}{\partial X_1}$ being bounded by M' in R (cf. Lipschitz conditions verified by E) one can choose the bounds $a_X, b_{x_0^\alpha}$ and C_{x^4} in such a way that, in v one has

$$\left|\frac{\partial E}{\partial Z_1}\right| \leq M' + \alpha',\ \alpha' \text{ being an arbitrarily small number.}$$

One can also choose the bounds $b_{x_0^\alpha}$ and C_{x^4} so that, in v, β being an arbitrarily small number, one has

$$|I\ E(X_1, z_0^\alpha, z^4)| \leq \beta,\ |R\ E(X_1, z_0^\alpha, z^4)| \leq M + \beta.$$

One can build on the other hand a cover of the domain R by means of a finite number of projections in R of the m previous cylinders, the corresponding m cylinders determine a domain $\overline{R}$ of the space of complex arguments Z, z_0, z^4, which fulfill the inequalities

$$|X - \overline{X}_0| \leq d,\ \left|x_0^i - \overline{x}^i\right| \leq d,\ |x^4| \leq |x_0^4| \leq \varepsilon(x_0^i),$$

$$|Y| \leq a,\ |y_0^\alpha| \leq b,\ |y^4| \leq c,$$

a, b, c being nonvanishing numbers, and in which the complex function E is defined and analytic. Let us write:

$$E(Z_1, z_0^\alpha, z^4) = E(Z_1, z_0^\alpha, z^4) - E(X_1, z_0^\alpha, z^4) + E(X_1, z_0^\alpha, z^4),$$

from which

$$|I\ E(Z_1, z_0^\alpha, z^4)| \leq m(M' + \alpha')a + \beta$$

$$|R\ E(Z_1, z_0^\alpha, z^4)| \leq m(M' + \alpha')a + \beta + M.$$

Let us consider now the equations (1), extended to the complex domain $\overline{R}$,

$$Z = \int_{z_0^4}^{z^4} E(Z, z_0^\alpha, z^4)dz^4 + Z_0. \qquad (\overline{1})$$

In order to solve it we consider, as in the real case, a functional space Υ defined by the functions of complex variables $Z_1(z_0^\alpha, z^4)$, real for z_0^α and z^4 real, analytic in the domain $\overline{D}$ defined by

$$\left|x_0^i - \overline{x}^i\right| \leq d,\ |x^4| \leq |x_0^4| \leq \varepsilon(x_0^i),\ |y_0^\alpha| \leq b,\ |y^4| \leq c,$$

and satisfying $|X_1 - X_0| \leq d,\ |y_1| \leq a$.

(1^o) The representation

$$Z_2 = \int_{z_0^4}^{z^4} E(Z_1, z_0^\alpha, z^4)dz^4 + Z_0$$

is a representation of the space into itself if $\varepsilon(x_0^i), b$ and c are suitably chosen. As a matter of fact:

(1) Z_1 is an analytic function of z_0^α, z^4 because this holds for E, real for z_0^α and z^4 real.

(2) From the equality

$$Z_2 = -\int_{x_0^4}^{x_0^4 + iy_0^4} E\ dz^4 + \int_{x_0^4}^{x^4} E\ dz^4 + \int_{x^4}^{x^4 + iy^4} E\ dz^4 + Z_0$$

we deduce

$$|X_2 - X_0| \leq (b + c)[m(M' + \alpha')a + \beta] + \left|x_0^4 - x^4\right| [m(M' + \alpha')a + \beta + M]$$

$$|Y_2| \le (b+c)[m(M'+\alpha')a+\beta+M] + \left|x_0^4 - x^4\right| [m(M'+\alpha')a+\beta.]$$

We shall thus have

$$|X_2 - \overline{X}_0| \le d \text{ if } \varepsilon(x_0^i) \le \frac{d-(b+c)[m(M'+\alpha')a+\beta]}{M+m(M'+\alpha')a+\beta}$$

and

$$|Y_2| \le a \text{ if } b+c \le \frac{a[1-mM'(x_0^4-x^4)]-(m\alpha' a+\beta)(x_0^4-x^4)}{M+m(M'+\alpha')a+\beta}.$$

Let us recall that the number

$$\varepsilon(x_0^i) < \frac{1}{mM}. \qquad (1)$$

We have therefore

$$1 - mM'(x_0^4 - x^4) > 0. \qquad (2)$$

We shall therefore choose $\varepsilon(x_0^i)$ as satisfying (1); the inequality (2) shows that one can find, without supplementary assumptions upon $\varepsilon(x_0^i)$, the numbers b and c defining $\overline{D}$ (after having chosen α', a and β sufficiently small), so that $\mathcal{M}_2$ is a point of $\mathcal{F}$. The domain $\overline{D}$ has for real part a domain as close as one wants to D.

(2^o) Let us prove that the representation reduces the distances. We have seen that, in $\overline{R}$, one has $\left|\frac{\partial E}{\partial Z_1}\right| \le M' + \alpha'$, from which

$$\left|E(Z_1', z_0^\alpha, z^4) - E(Z_1, z_0^\alpha, z^4)\right| \le |Z_1' - Z_1|(M'+\alpha');$$

we shall thus have

$$d(\mathcal{M}_2, \mathcal{M}_2') \le m(M'+\alpha')\left|z_0^4 - z^4\right| d(\mathcal{M}_1, \mathcal{M}_1'),$$

from which

$$d(\mathcal{M}_2, \mathcal{M}_2') \le d(\mathcal{M}_1, \mathcal{M}_1') \text{ if } \left|z_0^4 - z^4\right| < \frac{1}{mM'+\alpha'},$$

which will be in particular obtained if

$$\varepsilon(x_0^i) < \frac{1}{mM'+\alpha'} - \eta \text{ and } b+c < \eta,$$

η being an arbitrarily small number,

The real part of the domain $\overline{D}$ so defined is again as close as one wants to D.

We shall conclude, as in the real case, that the representation (I) admits a unique fixed point: the corresponding Z functions are solutions of equations (1), and analytic, in the domain $\overline{D}$. The functions X, values of these functions Z for real arguments x_0^4, x^4 are analytic functions, solutions in a domain as close as one wants to D of equations (1).

An analogous result is proved in the same way for equations (2), (3) and (4).

25 Coefficients and Cauchy data satisfying only the assumptions B and B'

If the coefficients $A^{\lambda\mu}$ and f_s, as well as the given functions $\overset{(1)}{W}_s$ and the Cauchy data, satisfy only the assumptions B and B' *we shall approach uniformly these quantities*, and at the same time their partial derivatives up to the fourth order, *by means of analytic functions*

$$A^{\lambda\mu}_{(n)}, f_{s(n)}, \overset{(1)}{W}_{s(n)}, \varphi_{s(n)}, \psi_{s(n)}$$

verifying, themselves as well, the assumptions B and B'.

We shall build in this way a family of functions $W_{s(n)}, ...U_{S(n)}$, solutions in D of equations $I_{1(n)}$ and solutions in D of the Cauchy problem $\left(\varphi_{s(n)}, \psi_{s(n)}\right)$, relatively to the equations $G_{1(n)}$:

$$A^{\lambda\mu}_{(n)} \frac{\partial^2 W_{s(n)}}{\partial x^\lambda \partial x^\mu} + f_{s(n)} = 0.$$

These functions $W_{s(n)}$ possess partial derivatives up to the fourth order and satisfy the same assumptions B and B' as is the case for the functions $\overset{(1)}{W}_s$.

26 Convergence of solutions of the approximate equations $G_{1(n)}$

Let us prove that these functions $\left(W_{s(n)}...U_{S(n)}\right)$ converge uniformly to some functions $(W_s...U_S)$ when the functions $A^{\lambda\mu}_{(n)}, \overset{(1)}{W}_{s(n)}, \varphi_{s(n)}, \psi_{s(n)}$ and their partial derivatives converge uniformly to the given functions

$$A^{\lambda\mu}, \overset{(1)}{W}_s, \varphi_s, \psi_s.$$

Arguments analogous to those of the previous pages, and the fact that the functions $W_{(n)}$ and $U_{(n)}$ verify a Lipschitz condition with respect to the x variables (that one has to replace by $X_{(n)}$ in the integral equations $(I_{1(n)})$ verified by these functions) show that

$$|X_{(n)} - X_{(m)}| \leq \mathrm{Max}_\Lambda \Big\{ \alpha \Big(\sum \left| A^{\lambda\mu}_{(n)} - A^{\lambda\mu}_{(m)} \right| + ...$$

$$+ \sum \left| \overset{(1)}{W}_{s(n)} - \overset{(1)}{W}_{s(m)} \right| + ... \Big)$$

$$+ M' \sum |X_{(n)} - X_{(m)}| \Big\} \left| x^4_0 - x^4 \right|,$$

$$|\Omega_{(n)} - \Omega_{(m)}| \leq \mathrm{Max}_\Lambda \Big\{ \beta \Big(\sum \left| A^{\lambda\mu}_{(n)} - A^{\lambda\mu}_{(m)} \right| + ... + \sum \left| \overset{(1)}{W}_{(n)} - \overset{(1)}{W}_m \right|$$

$$+ \sum |X_{(n)} - X_{(m)}| \Big) + N' \Big(|\Omega_{(n)} - \Omega_{(m)}|$$

$$+ \sum |W_{(n)} - W_{(m)}| \Big) \Big\} \left| x^4_0 - x^4 \right|,$$

$$\begin{aligned} |W_{(n)} - W_{(m)}| &\leq \mathrm{Max} \Big\{ \sum |W_{(n)} - W_{(m)}| + \sum |U_{(n)} - U_{(m)}| \Big\} |x^4| \\ &+ |W_{0(n)} - W_{0(m)}|, \end{aligned} \tag{26.1}$$

$$|U_{(n)} - U_{(m)}| \leq \mathrm{Max} \Big\{ \gamma \Big(\sum \left| A^{\lambda\mu}_{(n)} - A^{\lambda\mu}_{(m)} \right| + ... + \sum |f_{s(n)} - f_{s(m)}| + ...$$

$$+ \sum \left| \overset{(1)}{W}_{(n)} - \overset{(1)}{W}_m \right| + \sum |X_{(n)} - X_{(m)}| + R'_1 \Big(\sum |U_{(n)} - U_{(m)}|$$

$$+ \sum |W_{(n)} - W_{(m)}| + \sum |\Omega_{(n)} - \Omega_{(m)}| + \sum |\tilde{\Omega}_{(n)} - \tilde{\Omega}_{(m)}| \Big\} \cdot \left| x^4_0 \right|$$

$$+ \mathrm{Max} \Big\{ \delta \Big(\sum |X_{(n)} - X_{(m)}| + \sum \left| A^{\lambda\mu}_{(n)} - A^{\lambda\mu}_{(m)} \right| + ...$$

$$+\sum|\Phi_{s(n)}-\Phi_{s(m)}|\Big)+R_2'\sum|\Omega_{(n)}-\Omega_{(m)}|\Big\}_{x^4=0}.$$

$\alpha,\beta,\gamma,\delta$ are bounded numbers (which only depend on the bounds (B), (B'), h and h'.

The written inequalities show without difficulty that the functions

$$X_{(n)},\Omega_{(n)} \text{ and } W_{(n)},U_{(n)}$$

converge uniformly towards functions X,Ω and W,U in their respective domains of definition, (Λ) and (D),[10] when the approximating functions converge uniformly towards the given functions.

These functions W,U, uniform limits of the functions $W_{s(n)},U_{(n)}$, satisfy the following properties.

27 Properties of the solutions of equations G_1

(1^o) The functions $W_{s\alpha}...U_S$ are partial derivatives up to the fourth order of the functions W_s, and all these functions satisfy the same assumptions (B) and (B') as the functions $\overset{(1)}{W}_s$ in D.

(2^o) The functions W_s verify the partial differential equations G_1:

$$\overset{(1)}{A}{}^{\lambda\mu}\frac{\partial^2 W_s}{\partial x^\lambda\partial x^\mu}+f_s=0$$

in the domain D.

28 Solution of the given equations G

We consider the functional space W defined by the functions $\overset{(1)}{W}_s$ and satisfying the assumptions (B) and (B') in the domain D. We have just proved that the solution evaluated of the Cauchy problem for the equations G_1 defines a representation of this space into itself. Let us denote by $\overset{(1)}{W}_s$ this solution.

The space W is a normed, complete and compact space (for the topology of uniform convergence) if one defines the distance of two of its points by

$$d(\mathcal{M}_1,\mathcal{M}_1')=\mathrm{Max}_D\left(\sum\left|\overset{(1)}{W}_s-\overset{(1)'}{W}_s\right|+...+\left|\overset{(1)}{U}_S-\overset{(1)'}{U}_S\right|\right).$$

The distance of two representative points $\mathcal{M}_2,\mathcal{M}_2'$ from $\mathcal{M}_1,\mathcal{M}_1'$ will be compared to the distance of these points with the help of inequalities analogous to the inequalities (26.1) (the terms relative to the differences of the coefficients $A^{\lambda\mu}$, f_s and of the Cauchy data being suppressed).

It is then clear that there exists a number η bounded, nonvanishing, such that if the number $\varepsilon(x_0^i)$, defining the domain D, verifies

$$\varepsilon(x_0^i)<\eta,$$

the distance of the two representative points

$$\left(\overset{(2)'}{W}_s...\overset{(2)'}{U}_S\right) \text{ and } \left(\overset{(2)}{W}_s...\overset{(2)}{U}_S\right)$$

is less than the distance of the initial points.

[10] The number $\varepsilon(x_0^i)$ that defines D having been chosen in such a way that the representations defined with the help of these equations reduce the distances: $\varepsilon(x_0^i)<\frac{1}{m\,M}$ etc.

The representation considered admits then a unique fixed point $(W_s...U_S)$ which belongs to the space.

The functions W_s corresponding to this fixed point are solutions of the Cauchy problem, formulated in relation with the given equations G, in the domain D. They possess partial derivatives up to the fourth order, continuous, bounded and satisfying Lipschitz conditions with respect to the variables x^i.

We arrive also to the existence theorem that we state as follows.

29 Existence theorem

The Cauchy problem relative to the system of nonlinear partial differential equations

$$A^{\lambda\mu}(W_r)\frac{\partial^2 W_s}{\partial x^\lambda \partial x^\mu} + f_s(W_r, W_{r\lambda}) = 0 \; \lambda, \mu = 1, 2, 3, 4, \; s, r = 1, 2, ...n, \tag{G}$$

admits in the domain D, under the assumptions H, a solution possessing partial derivatives up to the fourth order, continuous and bounded and satisfying Lipschitz conditions with respect to the variables x^i.

30 Uniqueness theorem

Let us consider the system of integral equations verified by the solutions of the given equations G. This system can only have one solution $W_s, W_{s\alpha}, ..., U_S$ where the $W_{s\alpha}, ..., U_S$ are partial derivatives of the W_s: in this case there occurs indeed no difficulty in writing inequalities analogous to the inequalities of Sec. 26, where $W_{(n)}...U_{(n)}; \overset{(1)}{W}_{(n)}...\overset{(1)}{U}_{(n)}$ on the one hand,

$$W_{(m)}...U_{(m)}; \overset{(1)}{W}_{(m)}...\overset{(1)}{U}_{(m)}$$

on the other hand, would be replaced by two solutions of equations G, respectively; from these inequalities one derives without suffering the coincidence of these two solutions.

31 Summary of the results of chapter III

Let us summarize here the assumptions made and the results obtained. We consider a system of nonlinear, second-order, hyperbolic partial differential equations with n unknown functions W_s and four variables x^α, of the form

$$E_s = A^{\lambda\mu}\frac{\partial^2 W_s}{\partial x^\lambda \partial x^\mu} + f_s = 0, \; \lambda, \mu = 1, 2, 3, 4, \; s = 1, 2..., n. \tag{E}$$

The f_s are given functions of the unknown W_s, of their first partial derivatives $W_{s\alpha}$ and of the variables x^α. The $A^{\lambda\mu}$ are given functions of the W_s and of the x^α.

The Cauchy data are, on the initial surface $x^4 = 0$,

$$W_s(x^i, 0) = \varphi_s(x^i), \; W_{s4}(x^i, 0) = \psi_s(x^i).$$

On the system (E) and the Cauchy data I make the following assumptions:

(1^o) In the domain (d), defined by $|x^i - \overline{x}^i| \leq d$, φ_s and ψ_s possess partial derivatives up to the orders five and four, continuous, bounded and satisfying Lipschitz conditions.

(2^o) For the values of the W_s satisfying

$$|W_s - \varphi_s| \leq l, \; |W_{si} - \varphi_{si}| \leq l, \; |W_{s4} - \psi_s| \leq l$$

and in the domain D defined by

$$|x^i - \overline{x}^i| \leq d,\ |x^4| \leq \varepsilon :$$

(a) $A^{\lambda\mu}$ and f_s possess partial derivatives up to the fourth order, continuous, bounded and satisfing Lipschitz conditions.

(b) The quadratic form $A^{\lambda\mu}X_\lambda X_\mu$ is of the normal hyperbolic type: $A^{44} > 0$, $A^{ij}\xi_i\xi_j$ negative-definite.

I then prove that the Cauchy problem (φ_s, ψ_s) admits a unique solution, possessing partial derivatives continuous and bounded up to the fourth order, in relations with equations (E) in a domain $\triangle$ (tronc of cone with base d):

$$|x^i - \overline{x}^i| \leq d,\ |x^4| \leq \eta(x^i).$$

CHAPTER IV

Existence and uniqueness theorems for the equations of relativistic gravitation.

The ten potentials $g_{\alpha\beta}$ of the ds^2 of an Einstein universe satisfy, in the domains without matter and in absence of electromagnetic field, the ten partial differential equations of second order of the exterior case

$$R_{\alpha\beta} \equiv \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\lambda\beta} + \Gamma^\lambda_{\lambda\mu}\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\lambda\alpha}\Gamma^\lambda_{\mu\beta} = 0,$$

where one has set ∂_λ for $\frac{\partial}{\partial x^\lambda}$ and where the x^λ are a system of four spacetime coordinates whatsoever.

The ten equations are not independent because the $R_{\alpha\beta}$ satisfy the four conservation conditions (Bianchi identities)

$$\nabla_\lambda S^{\lambda\mu} \equiv 0 \text{ where } S^{\lambda\mu} \equiv R^{\lambda\mu} - \frac{1}{2}g^{\lambda\mu}R.$$

1 Cauchy problem

The problem of determinism, in the theory of relativistic gravitation, is formulated, for an exterior spacetime in the form of the Cauchy problem relative to the system of partial differential equations $R_{\alpha\beta} = 0$ and with initial data (potentials and first derivatives) carried by any hypersurface S.

The study of the values on S of the consecutive partial derivatives of the potentials has shown that, if S is nowhere tangent to a characteristic manifold, and if the Cauchy data satisfy four given conditions, the Cauchy problem admits, with respect to the system of equations $R_{\alpha\beta} = 0$, in the analytic case, a solution. This solution is unique, i.e., if there exist two solutions, they coincide up to a change of coordinates (conserving S pointwise and the values on S of the Cauchy data).

If S is defined by the equation $x^4 = 0$, the four conditions that the initial data must verify are the four equations

$$S^4_\lambda = 0$$

which are expressed in terms of the data only.

2 Isothermal coordinates

The coordinate x^λ is said to be isothermal if the potentials satisfy the following first-order partial differential equation:

$$F^\lambda \equiv \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\, g^{\lambda\mu})}{\partial x^\mu} = 0.$$

The Einstein equations read as, in whatever coordinates,

$$R_{\alpha\beta} \equiv -G_{\alpha\beta} - L_{\alpha\beta} = 0$$

with

$$G_{\alpha\beta} \equiv \frac{1}{2} g^{\lambda\mu} \frac{\partial^2 g_{\alpha\beta}}{\partial x^\lambda \partial x^\mu} + H_{\alpha\beta},$$

where $H_{\alpha\beta}$ is a polynomial of the $g_{\lambda\mu}, g^{\lambda\mu}$ and of their first derivatives and

$$L_{\alpha\beta} \equiv \frac{1}{2} g_{\beta\mu} \partial_\alpha F^\mu + \frac{1}{2} g_{\alpha\mu} \partial_\beta F^\mu. \tag{2.1}$$

We see that, if the four coordinates are isothermal, every equation $R_{\alpha\beta} = 0$ does not contain second derivatives besides those of $g_{\alpha\beta}$. The system of Einstein equations takes then the form of the systems studied in the previous chapters.

We can, without restricting the generality of the hypersurface S[11], assume that the initial data satisfy, besides the four conditions $S^4_\lambda = 0$, the conditions of isothermy:

$$F^\mu \equiv \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\, g^{\lambda\mu})}{\partial x^\lambda} = 0 \text{ for } x^4 = 0. \tag{2.2}$$

We shall solve this Cauchy problem for the equations $G_{\alpha\beta} = 0$, verified by the potentials in isothermal coordinates, and we shall prove afterwards that the potentials obtained define indeed a spacetime, related to isothermal coordinates, and verify the equations of gravitation $R_{\alpha\beta} = 0$.

3 Solution of the Cauchy problems for the equations $G_{\alpha\beta} = 0$

We shall apply to the system

$$G_{\alpha\beta} \equiv g^{\lambda\mu} \frac{\partial^2 g_{\alpha\beta}}{\partial x^\lambda \partial x^\mu} + H_{\alpha\beta} = 0$$

the results of chapter III by setting $g^{\lambda\mu} = A^{\lambda\mu}, H_{\alpha\beta} = f_s, g_{\alpha\beta} = W_s$. Let us make on the Cauchy data the following assumptions:

Assumptions

In a domain (d) of the initial surface $S, x^4 = 0$, defined by

$$|x^i - \overline{x}^i| \leq d :$$

(1^o) The Cauchy data φ_s and ψ_s possess partial derivatives continuous and bounded up to the orders five and four, respectively.

(2^o) In the domain (d) and for these Cauchy data the quadratic form $g^{\lambda\mu} X_\lambda X_\mu$ is of normal hyperbolic type, S being oriented in space: $g^{44} > 0$, $g^{ij} X_i X_j$ negative-definite (let us remark in particular that the determinant g of the $g_{\lambda\mu}$ is $\neq 0$).

We deduce from these assumptions the existence of a number l such that for $|g_{\alpha\beta} - \overline{\varphi}_s| \leq l$ one has $g \neq 0$ and we see that, for some unknown functions $g_{\alpha\beta} = W_s$, the inequalities

$$|W_s - \overline{\varphi}_s| \leq l, \ \left| \frac{\partial W_s}{\partial x^i} - \frac{\partial \overline{\varphi}_s}{\partial x^i} \right| \leq l, \ \left| \frac{\partial W_s}{\partial x^4} - \overline{\psi}_s \right| \leq l \tag{3.1}$$

are satisfied. The coefficients of the equations $G_{\alpha\beta} = 0$ (which are here independent of the variables x^α satisfy, as the Cauchy data, the assumptions of chapter III, i.e.:

(1^o) The coefficients $A^{\lambda\mu} = g^{\lambda\mu}$ and $f_s = H_{\alpha\beta}$ admit partial derivatives with respect to all their arguments up to the fourth order continuous and bounded and satisfying Lipschitz conditions ($g^{\lambda\mu}$

[11] Once a spacetime and an hypersurface $S(x^4 = 0)$ are given, there always exists a coordinate change $\tilde{x}^\lambda = f(x^\mu)$, with $\tilde{x}^4 = 0$ for $x^4 = 0$, such that the potentials $\breve{g}_{\alpha\beta}$ verify the conditions (2.2) (an hypersurface S can always be integrated in a family of isothermal manifolds). Cf. the proof of Sec. 5.

and $H_{\alpha\beta}$ are rational fractions with denominator g of the $g_{\lambda\mu} = W_s$, and of the $g_{\lambda\mu} = W_s$ and $\frac{\partial W_s}{\partial x^\alpha}$, respectively).

(2^o) The quadratic form $A^{\lambda\mu} X_\lambda X_\mu$ is of normal hyperbolic type: $A^{44} > 0$, $A^{ij} X_i X_j$ negative-definite.

We can thus apply to the system $G_{\alpha\beta} = 0$, for the Cauchy problem here considered, the conclusion of chapter II, which is stated as follows.

There exists a number $\varepsilon(x^i) \neq 0$ such that, in the domain

$$|x^i - \overline{x}^i| < d, \ |x^4| \leq \varepsilon(x^i)$$

the Cauchy problem relative to the equations $G_{\alpha\beta} = 0$ admits a solution which has partial derivatives continuous and bounded up to the fourth order and which verifies the inequalities (3.1).

4 The solution of the system $G_{\alpha\beta} = 0$ verifies the conditions of isothermy

(1^o) *The solution found of the system $G_{\alpha\beta} = 0$ verifies the four equations*

$$\partial_4 F^\mu = 0 \text{ for } x^4 = 0.$$

We have assumed indeed that the initial data satisfy the conditions

$$S^4_\lambda = 0 \tag{4.1}$$

and

$$F^\mu = 0 \tag{4.2}$$

for $x^4 = 0$. Hence we have

$$S^4_\lambda \equiv -g^{4\mu}\Big\{G_{\lambda\mu} - \frac{1}{2} g_{\lambda\mu} g^{\alpha\beta} G_{\alpha\beta} + L_{\lambda\mu} - \frac{1}{2} g_{\lambda\mu} g^{\alpha\beta} L_{\alpha\beta}\Big\}.$$

The solution of the system $G_{\alpha\beta} = 0$ verifies therefore, taking into account the expression (2.1) of $L_{\alpha\beta}$, the equations

$$-\frac{1}{2} g^{4\mu} g_{\lambda\alpha} \partial_\mu F^\alpha - \frac{1}{2} \partial_\lambda F^4 + \frac{1}{2} \delta^4_\lambda \partial_\alpha F^\alpha = 0 \text{ for } x^4 = 0,$$

from which, by virtue of (4.2) ($F^\mu = 0$ and $\partial_\lambda F^\mu = 0$),

$$-\frac{1}{2} g^{44} g_{\lambda\alpha} \partial_4 F^\alpha = 0 \text{ for } x^4 = 0.$$

We see eventually that the solution found of the system $G_{\alpha\beta} = 0$ verifies the four equations

$$\partial_4 F^\mu = 0 \text{ for } x^4 = 0.$$

(2^o) *The solution found of $G^{\alpha\beta} = 0$ verifies $F^\mu = 0$.*

This property is going to result from the conservation conditions. Ten potentials $g_{\alpha\beta}$ whatsoever satisfy indeed the four Bianchi identities $\nabla_\lambda \left(R^{\lambda\mu} - \frac{1}{2} g^{\lambda\mu} R\right) \equiv 0$, where $R^{\lambda\mu}$ is the Ricci tensor corresponding to these potentials.

A solution of the system $\sigma_{\alpha\beta} = 0$ verifies therefore the four equations

$$\nabla_\lambda \left(L^{\lambda\mu} - \frac{1}{2} g^{\lambda\mu} L\right) = 0,$$

where $L^{\lambda\mu} = g^{\alpha\lambda}g^{\beta\mu}L_{\alpha\beta}$ and $L = g^{\alpha\beta}L_{\alpha\beta}$. It turns out from the expression (2.1) of $L_{\alpha\beta}$ that these equations read as

$$\frac{1}{2}g^{\alpha\lambda}\nabla_\lambda(\partial_\alpha F^\mu) + \frac{1}{2}g^{\beta\mu}\nabla_\lambda(\partial_\beta F^\lambda) - \frac{1}{2}g^{\lambda\mu}\nabla_\lambda(\partial_\alpha F^\alpha) = 0,$$

from which, by developing and simplifying,

$$\frac{1}{2}g^{\alpha\lambda}\frac{\partial^2 F^\mu}{\partial x^\alpha \partial x^\lambda} + P_\mu(\partial_\alpha F^\lambda) = 0,$$

where P is a linear combination of the $\partial_\alpha F^\lambda$ whose coefficients are polynomials of the $g^{\alpha\beta}, g_{\alpha\beta}$ and of their first derivatives.

We notice therefore that the four quantities F^μ (formed with the $g_{\alpha\beta}$ solutions of $G_{\alpha\beta} = 0$) verify four partial differential equations of the type previously studied. The coefficients $A^{\lambda\mu} = g^{\lambda\mu}$ and $f_s = P_\mu$ verify, in D, the assumptions of chapter III. The quantities F^μ are by hypothesis vanishing on the domain (d) of $x^4 = 0$, and we have proved that the same was true of their first derivatives $\partial_\alpha F^\mu$. We deduce then from the uniqueness theorem that, in D, we have

$$F^\mu = 0 \text{ and } \partial_\alpha F^\mu = 0.$$

The potentials, solutions of the Cauchy problem formulated with respect to the system $G_{\alpha\beta} = 0$, verifies therefore effectively in (D) the conditions of isothermy and represent the potentials of an Einstein spacetime, solutions of the equations of gravitation $R_{\alpha\beta} = 0$.

5 Uniqueness

In order to prove that there exists only one exterior spacetime corresponding to the initial conditions given on S, one has to prove that every solution of the Cauchy problem formulated in such a way with respect to the equations $R_{\alpha\beta} = 0$ can be deduced by a change of coordinates from the solution of this Cauchy problem relative to the equations $G_{\alpha\beta} = 0$. We know (chapter IV) that this last solution is unique.

Let us therefore consider a solution $g_{\alpha\beta}$ of the Cauchy problem relative to the equations $R_{\alpha\beta} = 0$ and look for a transformation of coordinates

$$\check{x}^\alpha = f^\alpha(x^\beta).$$

By conserving S pointwise and in such a way that the potentials in the new system of coordinates, let them be $\check{g}_{\alpha\beta}$, verify the four equations

$$\check{F}^\lambda = 0,$$

we know that the four quantities $\check{F}^\lambda$ are invariants which verify the identities

$$\check{F}^\lambda \equiv \check{\triangle}_2 \check{x}^\lambda = \triangle_2 f^\lambda.$$

In order for the equations $\check{F}^\lambda = 0$ to be verified it is therefore necessary and sufficient that the functions f satisfy the equations

$$\triangle_2 f^\alpha \equiv g^{\lambda\mu}\left(\frac{\partial^2 f^\alpha}{\partial x^\lambda \partial x^\mu} - \Gamma^p_{\lambda\mu}\frac{\partial f^\alpha}{\partial x^p}\right) = 0 \tag{5.1}$$

which are partial differential equations of second order, linear, normal hyperbolic in the domain (D).

If we take for values of the functions f^α and of their first derivatives, upon S, the following values (that are such that the change of coordinates conserves S pointwise)

$$f^4 = 0,\ \partial_\alpha f^4 = \delta^4_\alpha,$$
$$f^i = x^i,\ \partial_\alpha f^i = \delta^i_\alpha, \tag{5.2}$$

for $x^4 = 0$, we see that the Cauchy problems formulated in such a way admit (cf. the existence theorems) in (D) and in relation to the equations (5.1) solutions possessing their partial derivatives up to the fourth order continuous and bounded.

We have thus defined a change of coordinates $\check{x}^\lambda = f^\lambda(x^\alpha)$ such that, in the new system of coordinates, the potentials $\check{g}_{\alpha\beta}$ verify the conditions of isothermy $\check{F}^\lambda = 0$. It remains to prove that this change of coordinates determines in a unique way the Cauchy data $\check{g}_{\alpha\beta}(x^4 = 0)$ and $\check{\partial}_4\check{g}_{\alpha\beta}(x^4 = 0)$, in terms of the original data $g_{\alpha\beta}(x^4 = 0)$ and $\partial_4 g_{\alpha\beta}(x^4 = 0)$.

We know that, $g_{\alpha\beta}$ being the components of a covariant two-index tensor

$$g_{\alpha\beta} = \check{g}_{\lambda\mu}\partial_\alpha f^\lambda \partial_\beta f^\mu, \tag{5.3}$$

from which, in light of (5.2),

$$g_{\alpha\beta} = \check{g}_{\alpha\beta} \qquad \partial_i g_{\alpha\beta} = \check{\partial}_i \check{g}_{\alpha\beta} \text{ for } x^4 = \check{x}^4 = 0.$$

It remains to evaluate the derivatives of the potentials with respect to x^4 and $\check{x}^4$ for $x^4 = \check{x}^4 = 0$. φ being an arbitrary function of a spacetime point we have

$$\partial_4\varphi = \check{\partial}_\lambda\varphi\partial_4 f^\lambda,$$

from which

$$\partial_4\varphi = \check{\partial}_4\varphi \text{ for } x^4 = \check{x}^4 = 0. \tag{5.4}$$

We find, on the other hand, by deriving the equality (5.3) with respect to x^4

$$\partial_4 g_{\alpha\beta} = \partial_4\check{g}_{\lambda\mu}\partial_\alpha f^\lambda \partial_\beta f^\mu + \check{g}_{\lambda\mu}\Big(\partial^2_{\alpha 4} f^\lambda \partial_\beta f^\mu + \partial^2_{\beta 4} f^\mu \partial_\alpha f^\lambda\Big),$$

from which

$$\partial_4 g_{\alpha\beta} = \partial_4\check{g}_{\alpha\beta} + \check{g}_{\lambda\beta}\partial^2_{\alpha 4} f^\lambda + \check{g}_{\mu\alpha}\partial^2_{\beta 4} f^\mu \text{ for } x^4 = 0. \tag{5.5}$$

We deduce also from the initial values (5.2) of the f^λ:

$$\partial^2_{\alpha i} f^\lambda = 0 \text{ for } x^4 = 0.$$

The f^λ verify on the other hand the conditions of isothermy (5.1), from which

$$g^{44}\partial^2_{44} f^\lambda = g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} \text{ for } x^4 = 0.$$

$\partial^2_{44} f^\lambda$ is hence determined in a unique way by the original Cauchy data; this is also equally true of $\partial_4\check{g}_{\alpha\beta}$ for $x^4 = 0$.

We have thus proved the following theorem:

Once a solution $g_{\alpha\beta}$ of the Cauchy problem is given in relation to the equations $R_{\alpha\beta} = 0$ (the initial data satisfying upon S the differentiability assumptions previously stated) there exists a change of coordinates, conserving S pointwise, such that the potentials $\check{g}_{\alpha\beta}$ in the new system of coordinates verify everywhere the conditions of isothermy and represent the solution, unique, of a Cauchy problem, determined in a unique way, relative to the equations $G_{\alpha\beta} = 0$.

We conclude therefore, in terms of relativity:

Theorem. There exists one and only one exterior spacetime corresponding to the initial conditions assigned upon S.

Acknowledgments

I am happy to express here my profound gratitude to M. Lichnerowicz who has made it possible for me to take advantage of the clarity he has in facing the great mathematical problems. After having suggested to undertake this work, he always provided his encouragement and advice, which made it possible for me to accomplish my task. I would also like to express my warmest thanks to M. G. Darmois: the benevolent interest that he has always shown for my work on problems of which he was the first to formulate the essential elements, has been precious for me.

To M. Leray, who wanted to join the examination committee of my thesis, and M. Peres, who accepted to play the role of president, I express my respectful gratitude.

I thank with great respect M. Marcel Riesz for his benevolent support, which has made it possible to publish this work.

References

[1] G. DARMOIS, *Les équations de la gravitation einsteinienne*, Mem. Sc. Math. **25**, Paris 1927.

[2] A. LICHNEROWICZ, *Problèmes globaux en mécanique relativiste*, Actual. Sci. Ind. **833**, Paris 1939.

[3] J. HADAMARD, *Le Problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, Paris 1932.

[4] M. RIESZ, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math. **81**, Uppsala 1949.

[5] Fl. BUREAU, *L'intégration des équations lineaires aux dérivées partielles du second ordre et du type hyperbolique normal*, Mem Soc. Royale Sc. de Liège, **3**, 1938.

[6] H. LEWY-K. FRIEDRICHS, *Das Anfangswertproblem einer beliebigen nichtlinearen hyperbolischen Differentialgleichung beliebiger Ordnung in zwei Variabeln. Existenz, Eindeutigkeit und Abhängigkeitsbereich der Lösung*, Math. Ann. **99**, Berlin 1928.

[7] J. SCHAUDER, *Das Anfangswertproblem einer quasilinearen hyperbolischen Differentialgleichung zweiter Ordnung in beliebiger Anzahl von unabhängigen Veränderlichen*, Fundam. Math. **24**, 1935.

[8] J. SCHAUDER, *Cauchysches Problem für partielle Differentialgleichungen erster Ordnung. Anwendung einiger sich auf Absolutbeträge der Lösungen beziehende Abschätzungen*, Comm. Math. Helv. **9**, 1936-37.

[9] G. HERGLOTZ, *Über die Integration linearer partieller Differentialgleichungen mit konstanten Koeffizienten*, Leipz. Ber. **80**, 1927.

[10] I. PETROWSKY, *Über das Cauchysche Problem über Systeme von partiellen Differentialgleichungen*, Rec. Math. Moscou, N. s. 2, 1937.

[11] S. L. SOBOLEV, *Méthode nouvelle a résoudre le problème de Cauchy pour les équations linéaires hyperboliques normales*, Rec. Math. Moscou, N. s. 1, 1936.

[12] K. STELLMACHER, *Zum Anfangswertproblem der Gravitationsgleichungen. Ausbreitungsgesetze für charakteristische Singularitäten der Gravitationsgleichungen*, Math. Ann. **115**, Berlin 1938.

[13] S. CHRISTIANOVICH, *Le problème de Cauchy pour les équations non linéaires hyperboliques*, Rec. Math. Moscou, N. s. 2, 1937.

MAX-PLANCK-INSTITUT FÜR WISSENSCHAFTSGESCHICHTE

Max Planck Institute for the History of Science

Preprints since 2013 (a full list can be found at our website)

437 Jürgen Renn **Schrödinger and the Genesis of Wave Mechanics**

438 Pietro Daniel Omodeo **L'*iter* europeo del matematico e medico scozzese Duncan Liddel**

439 Irina Tupikova & Klaus Geus **The Circumference of the Earth and Ptolemy's World Map**

440 Pietro Daniel Omodeo und Jürgen Renn **Das Prinzip Kontingenz in der Naturwissenschaft der Renaissance**

441 Horst Kant und Jürgen Renn **Eine utopische Episode – Carl Friedrich von Weizsäcker in den Netzwerken der Max-Planck-Gesellschaft**

442 William G. Boltz and Matthias Schemmel **The Language of 'Knowledge' and 'Space' in the Later Mohist Canon** (TOPOI – Towards a Historical Epistemology of Space)

443 Stefano Bordoni **Looking for a Rational Thermodynamics in the late XIX century**

444 Sonja Brentjes and Jürgen Renn **The Arabic Transmission of Knowledge on the Balance**

445 Horst Nowacki **Archimedes and Ship Design**

446 Matthias Schemmel **Elements of a Historical Epistemology of Space** (TOPOI – Towards a Historical Epistemology of Space)

447 Martin Thiering and Wulf Schiefenhövel **Spatial Concepts in Non-Literate Societies: Language and Practice in Eipo and Dene Chipewyan** (TOPOI – Towards a Historical Epistemology of Space)

448 Jürgen Renn **Einstein as a Missionary of Science**

449 Hubert Laitko **Der Ambivalenzbegriff in Carl Friedrich von Weizsäckers Starnberger Institutskonzept**

450 Stefano Bordoni **When Historiography met Epistemology.** Duhem's early philosophy of science in context

451 Renate Wahsner **Tausch – Allgemeines – Ontologie** oder **Das Auseinanderlegen des Konkreten und seine Aufhebung**

452 Jens Høyrup **Algebra in Cuneiform.** Introduction to an Old Babylonian Geometrical Technique

453 Horst Nowacki **Zur Vorgeschichte des Schiffbauversuchswesens**

454 Klaus Geus and Mark Geller (eds.) **Esoteric Knowledge in Antiquity** (TOPOI – Dahlem Seminar for the History of Ancient Sciences Vol. II)

455 Carola Sachse **Grundlagenforschung. Zur Historisierung eines wissenschaftspolitischen Ordnungsprinzips am Beispiel der Max-Planck-Gesellschaft (1945–1970)**

456 David E. Rowe and Robert Schulmann **General Relativity in the Context of Weimar Culture**

457 F. Jamil Ragep **From Tūn to Turun: The Twists and Turns of the Ṭūsī-Couple**

458 Pietro Daniel Omodeo **Efemeridi e critica all'astrologia tra filosofia naturale ed etica: La contesa tra Benedetti e Altavilla nel tardo Rinascimento torinese**

459 Simone Mammola **Il problema della grandezza della terra e dell'acqua negli scritti di Alessandro Piccolomini, Antonio Berga e G. B. Benedetti e la progressiva dissoluzione della cosmologia delle sfere elementari nel secondo '500**