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Zur Quantelung der Wellenfelder

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Einleitung

Wesentliche Fortschritte in der Formulierung der allgemeinen Quantengesetze der elektromagnetischen und materiellen Wellenfelder haben neuerdings Heisenberg und Pauli[1]) erzielt, indem sie die von Dirac erfundene „Methode der nochmaligen Quantelung" systematisch entwickelten. Neben gewissen sachlichen Schwierigkeiten, die viel tiefer liegen, trat dabei eine eigentümliche Schwierigkeit formaler Natur auf: der zum skalaren Potential kanonisch konjugierte Impuls verschwindet identisch, so daß die Aufstellung der Hamiltonschen Funktion und der Vertauschungsrelationen nicht ohne weiteres gelingt. Zur Beseitigung dieser Schwierigkeit sind bisher drei Methoden vorgeschlagen worden, die zwar ihren Zweck erfüllen, aber doch schwerlich als befriedigend betrachtet werden können.

1. Die erste Heisenberg-Paulische Methode ist ein rein analytischer Kunstgriff.[2]) Man fügt zur Lagrangefunktion gewisse Zusatzglieder hinzu, die mit einem kleinen Parameter ε multipliziert sind und bewirken, daß der obenerwähnte Impuls nicht mehr verschwindet. In den Schlußresultaten muß man dann zum Limes $\varepsilon = 0$ übergehen. Die ε-Glieder führen aber zu unphysikalischen Rechenkomplikationen[3]) und zerstören die charakteristische Invarianz der Lagrangefunktion gegenüber der Eichinvarianzgruppe.

2. Die zweite Heisenberg-Paulische Methode[4]) benutzt hingegen wesentlich diese Invarianz. Dem skalaren Potential

1) W. Heisenberg u. W. Pauli, Ztschr. f. Phys. **56**. S. 1. 1929; ebenda **59**. S. 168. 1930. Im folgenden mit H. P. I bzw. II zitiert.

2) H. P. I, S. 24—26, 30ff.

3) Vgl. L. Rosenfeld, Ztschr. f. Phys. **58**. S. 540. 1929.

4) H. P. II.

wird ein bestimmter, beliebiger Wert, z. B. Null, gegeben; dann liefert die Hamiltonsche Methode eine Bewegungsgleichung weniger. Lautet nun die fehlende Gleichung $C = 0$, so findet man auf Grund der Eichinvarianz der Hamiltonfunktion $C =$ konst. Die Wahl des Wertes 0 für diese Konstante bedeutet die Beschränkung auf eines von verschiedenen untereinander nicht kombinierenden Termsystemen. Das Auszeichnen einer Komponente des Viererpotentials bringt aber mit sich die Notwendigkeit eines Beweises für die relativistische Kovarianz des Verfahrens; und dieser Nachweis ist sehr mühsam.

3. Die Fermische Methode[1]) besteht auch im Hinzufügen von Zusatzgliedern zur Lagrangefunktion derart, daß kein Impuls mehr identisch verschwindet. Damit die so erhaltenen Feldgleichungen mit den gewöhnlichen übereinstimmen, müssen gewisse Nebenbedingungen erfüllt sein; es muß dann gezeigt werden, daß, wenn diese Nebenbedingungen auf einem Schnitt $t =$ const gelten, sie sich dann von selbst im Laufe der Zeit fortpflanzen. Der Nachteil dieser Methode ist der, daß wiederum die Eichinvarianz zerstört wird.

Nun ist das identische Verschwinden der genannten Impulskomponente keineswegs eine vereinzelte Erscheinung; denn der Grund dafür ist eben die Eichinvarianz der Lagrangefunktion, wie eine leichte, weiter unten ausführlich dargelegte Überlegung zeigt. Analoges, d. h. allgemeiner das Auftreten von identischen Relationen zwischen den Variablen und den konjugierten Impulsen, ist in allen Fällen zu erwarten, wenn die Lagrangefunktion eine geeignet gebaute Gruppe gestattet. Bei der näheren Untersuchung dieser Verhältnisse an Hand des besonders lehrreichen Beispieles der Gravitationstheorie, wurde ich nun von Prof. Pauli auf das Prinzip einer neuen Methode freundlichst hingewiesen, die es in durchaus einfacher und natürlicher Weise gestattet, das Hamiltonsche Verfahren beim Vorhandensein von Identitäten auszubilden, ohne den Nachteilen der bisherigen Methoden ausgesetzt zu sein. Im folgenden wird der Gegenstand zunächst vom all-

1) Vgl. H. P. II, S. 171, Fußnote.

gemeinen gruppentheoretischen Standpunkt behandelt, sodann an Hand verschiedener physikalischer Beispiele illustriert.[1])

Erster Teil: **Allgemeine Theorie**

§ 1. Ansätze über die Lagrange-Funktion und die zugrunde gelegte Gruppe

Wir betrachten irgendein dynamisches System, definiert durch Feldgrößen $Q_\alpha\,(x^1, x^2, x^3, x^4)$, welche von den Raumkoordinaten x^1, x^2, x^3 und von der Zeitkoordinate $x^4 = ct$ (und *nicht*, wie bei H. P., $x^4 = ict$!) abhängen. Über die Lagrangefunktion $\mathfrak{L}\,(Q; \partial Q / \partial x)$ brauchen wir keine Annahme zu machen, solange wir im Rahmen der klassischen Theorie bleiben, d. h. mit lauter c-Zahlen operieren; betrachten wir aber die Variabeln Q_α als q-Zahlen (während die Raumzeitkoordinaten immer c-Zahlen bleiben), so müssen wir berücksichtigen, daß der Satz von der Ableitung einer Funktionenfunktion seine allgemeine Gültigkeit verliert[2]); wollen wir also (und das wird der Fall sein) gewisse Eigenschaften der Lagrangefunktion, die aus diesem Satz fließen, beibehalten, so sind wir genötigt, über die Funktion $\mathfrak{L}$ solche einschränkenden Annahmen zu machen, daß die betreffenden Eigenschaften trotz des Versagens des genannten Satzes gültig bleiben. Es zeigt sich nun, daß diese Einschränkungen zwar vom mathematischen Standpunkt sehr weitgehend sein müssen, daß sie jedoch bei den physikalisch interessanten Lagrangefunktionen erfüllt sind. Sie betreffen einmal die analytische Beschaffenheit der Lagrangefunktion: diese soll höchstens quadratisch in den Ableitungen der Q_α sein; ferner die Reihenfolge der miteinander nicht vertauschbaren Größen.

Zur Abkürzung schreiben wir oft $Q_{\alpha,\nu}$ statt $\dfrac{\partial Q_\alpha}{\partial x^\nu}$, auch $\dot{Q}_\alpha$ statt $Q_{\alpha,4} \equiv \dfrac{\partial Q_\alpha}{\partial x^4}$. Ferner unterdrücken wir die Summa-

1) Hier möchte ich ein für allemal betonen, daß die in den Arbeiten H. P. I und II behandelten Spezialfälle mir oft den Weg zur gewünschten Verallgemeinerung zeigten. Es hätte wenig Zweck, im folgenden jedesmal darauf hinzuweisen.

2) Vgl. H. P. I, S. 18, ferner S. 14, Fußnote 1.

tionszeichen gemäß der bekannten Regel. Mit diesen Festsetzungen lautet nun unser Ansatz für die Lagrangefunktion:

$$(1) \qquad 2\mathfrak{L} = Q_{\alpha,\nu}\,\mathfrak{A}^{\alpha\nu,\,\beta\mu}(Q)\,Q_{\beta,\mu} + Q_{\alpha,\nu}\,\mathfrak{B}^{\alpha\nu}(Q) + \mathfrak{B}^{\alpha\nu}(Q)\,Q_{\alpha,\nu} + \mathfrak{C}(Q).$$

Obwohl nur die $\dot{Q}_\alpha$ mit den Q_α nicht vertauschbar sind, müssen wir doch auch für die anderen Ableitungen an einer bestimmten Reihenfolge festhalten; denn gewisse Operationen, z. B. d/dx^4, verwandeln die betreffenden Größen in andere, die untereinander nicht mehr vertauschbar sind, so daß das Resultat einer solchen Operation von der ursprünglichen Reihenfolge abhängt.

Da die c-Zahlüberlegungen oft an Allgemeinheit und Eleganz überlegen sind, wollen wir sie im folgenden zuweilen zur ersten Übersicht benutzen und nachher die für die q-Zahltheorie erforderlichen Modifikationen andeuten. Um aber unnötige Wiederholungen zu vermeiden, reden wir auch bei c-Zahlen von Vertauschungsrelationen, wobei wir natürlich die korrespondierenden Poissonschen Klammersymbole meinen.

Nun zur Definition der Transformationsgruppe, welche die Lagrangefunktion (in einem näher zu präzisierenden Sinne) gestatten soll. Es liegt uns in dieser Untersuchung keineswegs daran, die größtmögliche Allgemeinheit anzustreben, sondern die Darstellung nur so allgemein zu halten, daß in den physikalischen Anwendungen die tieferen Zusammenhänge klar hervortreten. Wir fragen also nicht nach der allgemeinsten Gruppe, die bei gegebener Lagrangefunktion Identitäten von der oben besprochenen Art zur Folge hat, sondern wir legen eine speziellere, wenn auch ausgedehnte, Klasse von kontinuierlichen unendlichen Gruppen zugrunde, von der wir zeigen, daß sie bei beliebiger Lagrangeschen (c-Zahl-)Funktion zu Identitäten führen.[1])

Wir charakterisieren unsere Gruppe durch ihre infinitesimale Transformation; wir nehmen an, daß sich sowohl die x^ν wie die Q_α auf bestimmte Weise transformieren, und zwar

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1) Die dabei benutzte Methode gibt übrigens sofort die Antwort auf die eben aufgeworfene allgemeine Frage. Bei speziellem Bau der Lagrangefunktion braucht die Gruppe sogar nicht unendlich zu sein, um Identitäten zu bedingen.

hängen die δx^ν bzw. δQ_α ab von r_0 willkürlichen reellen Funktionen $\xi^r(x)$ $(r = 1, 2, \ldots r_0)$ und ihren Ableitungen bis zur Ordnung k bzw. j; die Koeffizienten dieser Ableitungen sollen reell sein und (hierin liegt die Spezialisierung der Gruppen) in δx^ν nur von den x^ν, in δQ_α nur von den x^ν und den Q_α (und nicht von den Ableitungen der Q_α) abhängen. In Formeln: [1]

$$
(2)\quad \begin{cases} \delta x^\nu = a_r^{\nu,0}(x)\,\xi^r(x) + a_r^{\nu,\sigma}(x)\dfrac{\partial \xi^r}{\partial x^\sigma} + a_r^{\nu,\sigma\ldots\tau}(x)\dfrac{\partial^k \xi^r}{\partial x^\sigma \cdots \partial x^\tau}, \\ \delta Q_\alpha = c^0_{\alpha r}(x, Q)\,\xi^r(x) + c^\sigma_{\alpha r}(x, Q)\dfrac{\partial \xi^r}{\partial x^\sigma} \\ \qquad\qquad + c^{\sigma\ldots\tau}_{\alpha r}(x, Q)\dfrac{\partial^j \xi^r}{\partial x^\sigma \cdots \partial x^\tau}. \end{cases}
$$
[2]

Dazu kommt noch die wesentliche Voraussetzung, daß[1])

$$
(3)\qquad j \geqslant k + 1 .
$$

Was die Vertauschungseigenschaften der in (2) vorkommenden Funktionen betrifft, so sollen die ξ^r c-Zahlen sein und diese Eigenschaft bei allen Transformationen der Gruppe (2) behalten (wie es ja der Festsetzung, daß die ξ^r nur von den x^ν abhängen, entspricht). Da die a nur von den x^ν abhängen, dürfen wir sie gleichfalls als c-Zahlen betrachten. Dann sind auch die δx^ν c-Zahlen, wie es sein muß, damit wir die x^ν selber als c-Zahlen behandeln dürfen.

Die wichtigsten in der Physik vorkommenden Gruppen fallen unter diesen Typus (vgl. den zweiten Teil dieser Arbeit).

Es bleibt jetzt noch übrig, auszudrücken, daß das Integral

$$
\int \mathfrak{L}\, dx^1\, dx^2\, dx^3\, dx^4
$$

bei den Transformationen (2) invariant bleibt. Zu dem Zwecke führen wir zunächst einige Begriffe ein.

Neben die „lokale“ Variation $\delta \Phi\,(x, Q, \partial Q/\partial x, \ldots)$ tritt die „substantielle“ Variation [4]

$$
(4)\qquad \delta^* \Phi = \delta \Phi - \frac{d\Phi}{dx^\nu}\,\delta x^\nu ;
$$

1) Wir setzen $\dfrac{\partial^0 \xi}{(\partial x)^0} \equiv \xi$ und $\dfrac{\partial^{-1} \xi}{(\partial x)^{-1}} \equiv 0$. [3]

wenn wir die transformierten Größen mit einem Strich versehen, so ist

$$\delta \Phi = \Phi'[x'; Q'(x'); \ldots] - \Phi[x; Q(x); \ldots],$$

während

$$\delta^* \Phi = \Phi'[x; Q'(x); \ldots] - \Phi[x; Q(x); \ldots]$$

bedeutet. Daraus folgen unmittelbar die wichtigen, auch für q-Zahlen gültigen Formeln:

$$\delta^* \frac{d\Phi}{dx^\nu} = \frac{d}{dx^\nu} \delta^* \Phi, \tag{5}$$ [5]

$$\delta \frac{d\Phi}{dx^\nu} = \frac{d}{dx^\nu} \delta \Phi - \frac{d\Phi}{dx^\varrho} \frac{d\,\delta x^\varrho}{dx^\nu}. \tag{6}$$

Eine Größe $\mathfrak{K}$ heißt eine skalare Dichte (in bezug auf die Gruppe), wenn sie folgende Transformationseigenschaft hat:

$$\delta^* \mathfrak{K} + \frac{d}{dx^\nu} (\mathfrak{K}\,\delta x^\nu) = 0, \tag{7}$$

oder auch nach (4) [6]

$$\delta \mathfrak{K} + \mathfrak{K} \frac{d\,\delta x^\nu}{dx^\nu} = 0. \tag{8}$$

Größen hängen im allgemeinen von zweierlei Indizes ab: erstens von Indizes $\alpha, \beta, \gamma, \ldots$, deren Wertbereich derjenige vom Index α in Q_α ist, zweitens vom Indizes $\mu, \nu, \ldots$, die, wie der Index von x^ν, von 1 bis 4 laufen. Insbesondere vertritt der Index r von ξ^r ein oder mehrere Systeme von Indizes $(\alpha, \beta, \ldots; \mu, \nu, \ldots)$, die in einer beliebigen eindimensionalen Folge numeriert sind. Die Indizes von der Art $\alpha, \beta, \ldots$ können auch ihrerseits mehrfach sein und insbesondere Systeme von Indizes $\mu, \nu, \ldots$ enthalten.

Ein kontravarianter Tensor $K^{\alpha\nu}$ wird definiert durch die Transformationseigenschaft

$$\delta K^{\alpha\nu} = K^{\alpha\mu} \frac{d\,\delta x^\nu}{dx^\mu} - \underline{K^{\beta\nu} \frac{\partial\,\delta Q_\beta}{\partial Q_\alpha}}; \tag{9}$$

im Falle der q-Zahlen enthält diese Erklärung wegen des unterstrichenen Gliedes eine Willkür, welche wir durch die Festsetzung

$$\underline{K^{\beta\nu} \frac{\partial\,\delta Q_\beta}{\partial Q_\alpha}} = \frac{1}{2} \left(K^{\beta\nu} \frac{\partial\,\delta Q_\beta}{\partial Q_\alpha} + \frac{\partial\,\delta Q_\beta}{\partial Q_\alpha} K^{\beta\nu\dagger} \right)$$

beseitigen; dabei bezeichnet $x^\dagger$ das hermitisch konjugierte (adjungierte) von x. [Im folgenden gebrauchen wir allgemein die Bezeichnung

$$\underline{x} = \tfrac{1}{2}(x + x^\dagger).]$$

Durch diese Festsetzung bleibt ein hermitischer Tensor nach einer beliebigen Transformation der Gruppe hermitisch.

Ein kovarianter Tensor $K_{\alpha\nu}$ hat die Transformationseigenschaft:

$$\delta K_{\alpha\nu} = - K_{\alpha\mu} \frac{d\,\delta x^\mu}{dx^\nu} + \underline{K_{\beta\nu} \frac{\partial\,\delta Q_\alpha}{\partial Q_\beta}}; \tag{10}$$

analog zu (9) und (10) wird die Variation des gemischten Tensors $K_{\alpha\beta\ldots}{}^{\gamma\delta\ldots}{}_{\mu\nu\ldots}{}^{\pi\varrho\ldots}$ gebildet.

Eine Tensordichte $\mathfrak{K}^{\alpha\nu}$ transformiert sich wie das Produkt eines Tensors $K^{\alpha\nu}$ mit einer skalaren Dichte $\mathfrak{K}$, also:

$$\delta \mathfrak{K}^{\alpha\nu} = \mathfrak{K}^{\alpha\mu} \frac{d\,\delta x^\nu}{dx^\mu} - \underline{\mathfrak{K}^{\beta\nu} \frac{\partial\,\delta Q_\beta}{\partial Q_\alpha}} - \mathfrak{K}^{\alpha\nu} \frac{d\,\delta x^\mu}{dx^\mu}. \tag{11}$$

Jetzt sind wir imstande, die Invarianzforderung bezüglich der Lagrangefunktion $\mathfrak{L}$ zu formulieren. Damit nämlich das Integral $\int \mathfrak{L}\, dx^1\, dx^2\, dx^3\, dx^4$ invariant ist, soll nach bekannten Schlüssen [1]) $\mathfrak{L}$ bis auf eine Divergenz $\mathfrak{L}' \equiv \frac{d\mathfrak{K}^\nu}{dx^\nu}$ eine skalare Dichte sein. In Formeln:

$$\delta(\mathfrak{L} + \mathfrak{L}') + (\mathfrak{L} + \mathfrak{L}') \frac{d\,\delta x^\nu}{dx^\nu} = 0. \tag{12}$$

Da es uns, wie gesagt, nicht auf die größte Allgemeinheit ankommt, wollen wir uns damit begnügen, der Reihe nach folgende chrakteristische Fälle zu behandeln:

1° $\mathfrak{L}' = 0$, d. h. $\mathfrak{L}$ ist selber eine skalare Dichte:

$$\delta\mathfrak{L} + \mathfrak{L} \frac{d\,\delta x^\nu}{dx^\nu} = 0; \tag{13}$$

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1) Vgl. etwa E. Noether, Gött. Nachr. 1918. S. 211. — Die Divergenz $\frac{d\,\mathfrak{K}^\nu}{d\,x^\nu}$ tritt dann auf, wenn das Integral $\int \mathfrak{L}\; dx^1 \ldots dx^4$ nicht bei *beliebigem* Integrationsgebiet invariant ist, sondern nur wenn die $\mathfrak{K}^\nu$ am Rande verschwinden.

2° $\mathfrak{L}'$ enthält die zweiten Ableitungen

$$Q_{\alpha,\nu\varrho} \equiv \frac{\partial^2 Q_\alpha}{\partial x^\nu \, \partial x^\varrho}$$

nur linear, d. h.

$$\mathfrak{L}' \equiv \frac{d}{dx^\nu} [\mathfrak{f}^{\nu,\alpha\varrho}(Q)\, Q_{\alpha,\varrho}] \tag{14}$$

und es ist $j = 0$ [vgl. Formel (3)].

In beiden Fällen zerfällt die Untersuchung in zwei Schritte: a) Durchführung des erweiterten Hamiltonschen Verfahrens; b) Beweis der Kovarianz desselben bezüglich der betrachteten Gruppe.

Wir beginnen mit dem ersten Falle.

§ 2. Die konjugierten Impulse und die Identitäten

Von jetzt an legen wir also die Forderung (13) zugrunde.

Wir setzen zunächst

$$\mathfrak{P}^{\alpha\nu} = \frac{\partial \mathfrak{L}}{\partial Q_{\alpha,\nu}} \tag{15}$$

und nehmen als Impulse

$$\mathfrak{P}^\alpha \equiv \mathfrak{P}^{\alpha 4} = \frac{\partial \mathfrak{L}}{\partial \dot{Q}_\alpha}\,. \tag{16}$$

Beschränken wir uns zuerst auf die klassische (c-Zahl-) Theorie.

Ersetzen wir in (13) die δQ_α, $Q_{\alpha,\nu}$ und δx^ν durch ihre Werte (2), (6) als Funktionen der ξ^r und Ableitungen, so bekommen wir mehrere Identitäten, indem wir ausdrücken, daß die Koeffizienten der einzelnen Ableitungen von ξ^r identisch verschwinden sollen. Diese Identitäten enthalten aber im allgemeinen die $\dot{Q}_\alpha$ nicht nur durch die soeben eingeführten Funktionen $\mathfrak{P}^\alpha$, sondern auch in anderen Verbindungen (z. B. durch die anderen $\mathfrak{P}^{\alpha\nu}$, $\nu \neq 4$); für die Auflösung des Gleichungssystems (16) nach den $\dot{Q}_\alpha$ bieten sie also kein Interesse: sie stellen einfach Beziehungen dar, die jede Lösung $\dot{Q}_\alpha(Q, \mathfrak{P})$ dieses Systems von selbst erfülllt. Wesentlich anders liegen aber die Verhältnisse, wenn einige der betrachteten Identitäten nur die Q_α (nebst räumlichen Ableitungen) und die $\mathfrak{P}^\alpha$ enthalten: sie bedeuten dann, daß die Gleichungen (16) nicht alle voneinander unabhängig sind, so daß die allgemeine Lösung von gewissen willkürlichen Parametern (genauer: Raumzeitfunktionen) abhängt.

Nun tritt der letztere Fall bei der Gruppe (2) immer auf. Die höchsten in (13) vorkommenden Ableitungen von ξ^r sind die

$$\frac{\partial^{j+1}\,\xi^r}{\partial x^\sigma \cdots \partial x^\tau\, dx^\nu};$$

nach der Voraussetzung (3) lauten die entsprechenden Identitäten

(17c) [1] $$\sum{}' \mathfrak{P}^{\alpha\nu}\, c^{\sigma\ldots\tau}_{\alpha r} \equiv 0,$$

wobei die Summe sich über alle Permutationen der Zahlen $\nu, \sigma, \ldots, \tau$ erstreckt. Für $\nu = \sigma = \ldots \tau = 4$ hat man insbesondere

(18c) $$\mathfrak{P}^{\alpha}\, c^{44\ldots4}_{\alpha r} \equiv 0:$$

da nun die c nur die Q_α enthalten, haben wir in (18c) r_0 Identitäten von der zuletzt besprochenen Form vor uns, welche wir „eigentliche“ Identitäten nennen wollen. Es ist ferner leicht [7]
einzusehen, daß im allgemeinen (d. h. falls die Lagrangefunktion keine spezielleren Eigenschaften besitzt) keine weiteren eigentlichen Identitäten vorkommen. Die allgemeinste Lösung $\dot{Q}_\alpha(Q, \mathfrak{P}, \lambda)$ von (16) hängt also von r_0 willkürlichen Parametern λ ab.

In den bisherigen, in der Einleitung erwähnten Methoden half man sich entweder durch Zerstören der Invarianzeigenschaft der Lagrangefunktion (1. und 3. Methode) oder durch Auszeichnung einer speziellen Lösung $\dot{Q}_\alpha(Q, \mathfrak{P}, \lambda^0)$ (2. Methode). Im Gegensatz dazu ist der Grundgedanke der neuen Methode der, die Hamiltonsche Funktion in der üblichen Weise mittels der allgemeinen Lösung $\dot{Q}_\alpha(Q, \mathfrak{P}, \lambda)$ mit unbestimmten λ^r zu konstruieren, ohne sich zunächst um die eigentlichen Identitäten zu kümmern: Feldgleichungen und Vertauschungsrelationen haben die kanonische Form, die ersteren enthalten die λ^r. Zu diesem kanonischen Schema kommen schließlich die eigentlichen Identitäten als Nebenbedingungen hinzu. Wir werden

1) Den Nummern der Formeln, die nur für c-Zahlen unbeschränkte Gültigkeit besitzen, wird der Buchstabe c angehängt.

sehen, daß diese Methode außer ihrer Einfachheit noch den großen Vorteil hat, daß der Kovarianzbeweis des Verfahrens ohne Schwierigkeit durchführbar ist.

§ 3. Übergang zu den q-Zahlen

Zuvor müssen wir untersuchen, wie sich beim Übergang zu den q-Zahlen die eben geschilderten Verhältnisse gestalten. Nach (1) lautet dann (15):

$$\mathfrak{P}^{\alpha\nu} = \tfrac{1}{2}(\mathfrak{p}^{\alpha\nu} + \mathfrak{p}^{\alpha\nu\dagger}) = \underline{\mathfrak{p}^{\alpha\nu}}, \tag{19}$$

mit

$$\mathfrak{p}^{\alpha\nu} = \mathfrak{A}^{\alpha\nu;\,\beta\mu} Q_{\beta,\mu} + \mathfrak{B}^{\alpha\nu}. \tag{20}$$

Durch einen Strich über einen Index von der Art $\mu : \bar{\mu}$ deuten wir an, daß er nur von 1 bis 3 läuft; für überstrichene Indizes soll die Regel vom Weglassen des Summenzeichens ebenfalls gelten. Mit dieser Bezeichnung schreiben wir nach (19) und (20)

$$\begin{cases} \mathfrak{P}^{\alpha} = \underline{\mathfrak{p}^{\alpha}}, \\ \mathfrak{p}^{\alpha} = \mathfrak{A}^{\alpha\beta}\dot{Q}_{\beta} + \mathfrak{D}^{\alpha}, \end{cases} \tag{21}$$

wobei

$$\begin{cases} \mathfrak{A}^{\alpha\beta} \equiv \mathfrak{A}^{\alpha 4;\,\beta 4} \\ \mathfrak{D}^{\alpha} \equiv \mathfrak{A}^{\alpha 4;\,\beta\bar{\mu}} Q_{\beta,\bar{\mu}} + \mathfrak{B}^{\alpha 4} \end{cases} \tag{22}$$

gesetzt ist. Es ist in diesen Formeln

$$\mathfrak{A}^{\alpha\nu;\,\beta\mu} = \mathfrak{A}^{\beta\mu;\,\alpha\nu},$$

insbesondere

$$\mathfrak{A}^{\alpha\beta} = \mathfrak{A}^{\beta\alpha},$$

angenommen worden, was natürlich keine Einschränkung bedeutet.

Die Überlegung des vorigen Paragraphen liefert jetzt statt (17c) und (18c)

$$\sum \underline{c_{\alpha r}^{\sigma \dots \tau}\, \mathfrak{p}^{\alpha\nu}} \equiv 0 \tag{23}$$

und

$$\underline{c_{\alpha r}^{4 \dots 4}\, \mathfrak{p}^{\alpha}} \equiv 0. \tag{24}$$

Da insbesondere (24) in den $\dot{Q}_{\alpha}$ identisch gilt, so ist nach (21)

$$c_{\alpha r}^{4 \dots 4}\, \mathfrak{A}^{\alpha\beta} = 0, \tag{25}$$

$$c_{\alpha r}^{4 \dots 4}\, \mathfrak{D}^{\alpha} = 0: \tag{26}$$

die Koeffizienten der Lagrangefunktion müssen u. a. diese Beziehungen erfüllen, damit $\mathfrak{L}$ die verlangte Dichteeigenschaft haben kann. Die Relationen (25) und (26) heben wir für späteren Gebrauch hervor.

Nun aber können wir nicht weiterkommen, ohne etwas über die Vertauschungsrelationen $[Q_\alpha, \dot{Q}_\beta]$ zu wissen. Wenn wir die $\dot{Q}_\beta$ als Funktionen der Q_α und $\mathfrak{P}^\alpha$ kennen würden, so könnten wir den Wert von $[Q_\alpha, \dot{Q}_\beta]$ aus den kanonischen Vertauschungsrelationen, die wir, wie gesagt, beizubehalten wünschen, ableiten. Es ist indessen nicht einmal von vornherein sicher, ob wir aus (21) die $\dot{Q}_\beta$ als Funktionen der *Matrizen* $\mathfrak{P}^\alpha$ ableiten können, oder nur als Funktionen der *Matrixelemente* von $\mathfrak{P}^\alpha$. Der einzige Ausweg ist der, daß wir versuchsweise eine *Annahme* über $[Q_\alpha, \dot{Q}_\beta]$ machen, auf Grund deren die Lösung von (21) die Gestalt $\dot{Q}_\alpha(Q, \mathfrak{P}, \lambda)$ annimmt und nachher prüfen, ob die gemachte Annahme mit den kanonischen Vertauschungsrelationen verträglich ist.

Eine naheliegende Annahme ist folgende: die $[Q_\alpha, \dot{Q}_\beta]$ sollen *schiefe* Funktionen[1]) von den Q_α und $Q_{\alpha,\bar{\nu}}$, nicht aber von den $\dot{Q}_\alpha$ (bzw. den $\mathfrak{P}^\alpha$) sein. (Ob dabei, wenn Q_α und $\dot{Q}_\beta$ [8]
im selben Punkt genommen sind, unbestimmte Faktoren, wie $\delta(0)$, vorkommen, ist gleichgültig). Führen wir einige unmittelbare Folgerungen dieser Annahme an:

1. Nach (20) sind ebenfalls die $[Q_\alpha, \mathfrak{p}^{\beta\nu}]$ und $[Q_\alpha, \mathfrak{p}^{\beta\nu\dagger}]$ schiefe Funktionen der Q_α und $Q_{\alpha,\bar{\nu}}$ allein.

2. Die $[Q_\alpha, \dot{Q}_\beta]$, $[Q_\alpha, \mathfrak{p}^{\beta\nu}]$ und $[Q_\alpha, \mathfrak{p}^{\beta\nu\dagger}]$ sind mit jeder Funktion der Q_α und $Q_{\alpha,\bar{\nu}}$ vertauschbar.

3. Es ist

$$[Q_\alpha, \mathfrak{p}^{\beta\nu}] = [Q_\alpha, \mathfrak{p}^{\beta\nu\dagger}]. \tag{27}$$

Infolgedessen kann man statt (23) und (24)

$$\sum \underline{c^{\sigma\ldots\tau}_{\alpha r}\,\mathfrak{P}^{\alpha\nu}} = 0, \tag{28}$$

$$\mathfrak{F}_r \equiv \underline{c^{4\ldots 4}_{\alpha r}\,\mathfrak{P}^{\alpha}} = 0 \tag{29}$$

schreiben.

Aus (25) folgt nun, daß die N linearen Gleichungen

$$\mathfrak{A}^{\alpha\beta}\dot{Q}_\beta + \dot{Q}_\beta\mathfrak{A}^{\beta\alpha} = 2(\mathfrak{P}^\alpha - \mathfrak{D}^\alpha) \tag{21}$$

---

1) Eine q-Zahl x heißt *schief*, wenn $x^\dagger = -x$.

nicht alle unabhängig sind, sondern daß ihre Determinante $|\mathfrak{A}^{\alpha\beta}|$ den Rang $N - r_0$ hat. Da sie symmetrisch ist, gibt es einen [9] von Null verschiedenen Hauptminor vom Grade $N - r_0$; die dazu bezüglichen Indizes wollen wir mit einem Strich versehen:

$$|\mathfrak{A}^{\alpha'\beta'}| \neq 0,$$

während die übrigen doppelt gestrichen seien: α'', β'', $\cdots$. Die Determinante $|\mathfrak{A}^{\alpha'\beta'}|$, sowie ihre reziproke $|\mathfrak{A}_{\alpha'\beta'}|$, sind symmetrisch und es gilt:

$$\mathfrak{A}^{\alpha'\beta'}\mathfrak{A}_{\beta'\gamma'} = \delta^{\alpha'}_{\gamma'}, \tag{30}$$

wobei δ^{α}_{β} wie üblich gleich 0 oder 1 ist, je nachdem $\alpha \neq \beta$ oder $\alpha = \beta$.

Gelingt es also, eine *spezielle* Lösung $\dot{Q}_\beta{}^0(Q, \mathfrak{P})$ von (21) zu finden, so hat die allgemeinste Lösung die Form:

$$\dot{Q}_\beta = \dot{Q}_\beta{}^0 + \lambda^r x_{\beta r},$$

wo die λ^r r_0 willkürliche Parameter und $x_{\beta r}$ r_0 unabhängige Lösungen der homogenen Gleichungen

$$\mathfrak{A}^{\alpha\beta} x_{\beta r} + x_{\beta r}\mathfrak{A}^{\beta\alpha} = 0$$

darstellen. Nach (25) können wir nun

$$x_{\beta r} = c^{4\cdots 4}_{\beta r}$$

wählen und schreiben:

$$\dot{Q}_\beta = \dot{Q}_\beta{}^0 + \lambda^r c^{4\cdots 4}_{\beta r}. \tag{31}$$

Ferner behaupte ich, daß

$$\begin{cases} \dot{Q}^0_{\beta'} = \frac{1}{2}\{\mathfrak{A}_{\beta'\gamma'}(\mathfrak{P}^{\gamma'} - \mathfrak{D}^{\gamma'}) + (\mathfrak{P}^{\gamma'} - \mathfrak{D}^{\gamma'})\mathfrak{A}_{\gamma'\beta'}\} \\ \dot{Q}^0_{\beta''} = 0 \end{cases} \tag{32}$$

eine spezielle Lösung von (21) ist; ist dies nachgewiesen, so ist uns die Auflösung von (21) nach den $\dot{Q}_\beta$ wirklich gelungen: denn die Lösung (31) hat offenbar die verlangte Eigenschaft, daß vermöge der kanonischen Vertauschungsrelationen $[Q_\alpha, \dot{Q}_\beta]$ eine schiefe Funktion der Q_α und $Q_{\alpha,\bar{\nu}}$ wird.

Durch Einsetzen von (32) in die linke Seite von (21), [10] die wir für einen Augenblick $\mathfrak{T}_\alpha$ nennen wollen, bekommt man

$$
\begin{aligned}
\mathfrak{T}^{\alpha} &= \tfrac{1}{2}\,\mathfrak{A}^{\alpha\beta'}\{\mathfrak{A}_{\beta'\gamma'}(\mathfrak{P}^{\gamma'}-\mathfrak{D}^{\gamma'}) + (\mathfrak{P}^{\gamma'}-\mathfrak{D}^{\gamma'})\,\mathfrak{A}_{\gamma'\beta'}\} \\
&\quad + \tfrac{1}{2}\{\mathfrak{A}_{\beta'\gamma'}(\mathfrak{P}^{\gamma'}-\mathfrak{D}^{\gamma'}) + (\mathfrak{P}^{\gamma'}-\mathfrak{D}^{\gamma'})\,\mathfrak{A}_{\gamma'\beta'}\}\,\mathfrak{A}^{\beta'\alpha} \\
&= \mathfrak{A}^{\alpha\beta'}\mathfrak{A}_{\beta'\gamma'}(\mathfrak{P}^{\gamma'}-\mathfrak{D}^{\gamma'}) + (\mathfrak{P}^{\gamma'}-\mathfrak{D}^{\gamma'})\,\mathfrak{A}_{\gamma'\beta'}\mathfrak{A}^{\beta'\alpha} \\
&\quad + \tfrac{1}{2}\,\mathfrak{A}^{\alpha\beta'}[\mathfrak{P}^{\gamma'}-\mathfrak{D}^{\gamma'},\ \mathfrak{A}_{\gamma'\beta'}] + \tfrac{1}{2}[\mathfrak{A}_{\beta'\gamma'},\ \mathfrak{P}^{\gamma'}-\mathfrak{D}^{\gamma'}]\,\mathfrak{A}^{\beta'\alpha} \\
&= \mathfrak{A}^{\alpha\beta'}\mathfrak{A}_{\beta'\gamma'}(\mathfrak{P}^{\gamma'}-\mathfrak{D}^{\gamma'}) + (\mathfrak{P}^{\gamma'}-\mathfrak{D}^{\gamma'})\,\mathfrak{A}_{\gamma'\beta'}\mathfrak{A}^{\beta'\alpha},
\end{aligned}
$$

wegen der Folgerung 2 aus unserer Annahme. Für $\alpha = \alpha'$ [11]
ist bereits nach (30)

$$\mathfrak{T}^{\alpha'} = 2\,(\mathfrak{P}^{\alpha'} - \mathfrak{D}^{\alpha'}).$$

Nun sind nach der Theorie der linearen Gleichungen und unter Benutzung unserer Annahme über $[Q_\alpha, \dot{Q}_\beta]$ die Identitäten (29) äquivalent mit

$$\mathfrak{P}^{\alpha''} = \underline{\mathfrak{A}^{\alpha''\beta'}\mathfrak{A}_{\beta'\gamma'}\mathfrak{P}^{\gamma'}}$$

und ebenso (26) äquivalent mit

$$\mathfrak{D}^{\alpha''} = \mathfrak{A}^{\alpha''\beta'}\mathfrak{A}_{\beta'\gamma'}\mathfrak{D}^{\gamma'};$$

folglich ist auch

$$\mathfrak{T}^{\alpha''} = 2\,(\mathfrak{P}^{\alpha''} - \mathfrak{D}^{\alpha''}),$$

womit der Nachweis, daß (31), (32) die allgemeinste Lösung von (21) im Einklang mit den kanonischen Vertauschungsrelationen darstellt, vollständig erbracht ist. [12]

§ 4. Aufstellung der Hamiltonfunktion

Klassisch lautet die Hamiltonfunktion

$$\mathfrak{H} = \mathfrak{P}^{\alpha}\dot{Q}_{\alpha} - \mathfrak{L};$$

von jedem quantenmechanischen Ansatz müssen wir nun verlangen, daß

$$\frac{\partial\,\mathfrak{H}}{\partial\,Q_{\alpha,\bar{\nu}}} = -\,\frac{\partial\,\mathfrak{L}}{\partial\,Q_{\alpha,\bar{\nu}}}, \tag{33}$$

eine Eigenschaft, die sich für die Durchführung der Theorie als unentbehrlich erweisen wird. [13]

Nun ist

$$\left(\frac{\partial\,\mathfrak{L}}{\partial\,Q_{\alpha,\bar{\nu}}}\right)_{\mathfrak{P}^{\alpha}} = \left(\frac{\partial\,\mathfrak{L}}{\partial\,Q_{\alpha,\bar{\nu}}}\right)_{\dot{Q}_{\alpha}} + \underline{\left(\frac{\partial\,\dot{Q}_{\beta}}{\partial\,Q_{\alpha,\bar{\nu}}}\right)_{\mathfrak{P}^{\alpha}}\cdot\,\mathfrak{p}^{\beta}}$$

und nach (31), (32) enthält $\left(\frac{\partial \dot{Q}_\beta}{\partial Q_{\alpha,\bar{\nu}}}\right)_{\mathfrak{P}^\alpha}$ nicht mehr die $\mathfrak{P}^\alpha$ folglich dürfen wir schreiben:

$$\left(\frac{\partial \mathfrak{L}}{\partial Q_{\alpha,\bar{\nu}}}\right)_{\mathfrak{P}^\alpha} = \left(\frac{\partial \mathfrak{L}}{\partial Q_{\alpha,\bar{\nu}}}\right)_{\dot{Q}_\alpha} + \underline{\left(\frac{\partial \dot{Q}_\beta}{\partial Q_{\alpha,\bar{\nu}}}\right)_{\mathfrak{P}^\alpha} \cdot \mathfrak{P}^\beta}.$$

Die gewünschte Eigenschaft (33) hat also der Ansatz

(34) $$\mathfrak{H} = \underline{\dot{Q}_\alpha \mathfrak{P}^\alpha} - \mathfrak{L}.$$

Da nach (25) und (26)

$$\mathfrak{L}[Q; \dot{Q}(Q, \mathfrak{P}, \lambda)] = \mathfrak{L}(Q, \dot{Q}^0)$$

ist, können wir schreiben gemäß der Bezeichnung (29)

(35) $$\mathfrak{H} = \mathfrak{H}_0 + \lambda^r \mathfrak{F}_r,$$

mit [14]

(36) $$\mathfrak{H}_0 = \underline{\dot{Q}_\alpha{}^0 \mathfrak{P}^\alpha} - \mathfrak{L}[Q, \dot{Q}^0(Q, \mathfrak{P})].$$

Nun setzen wir die kanonischen Vertauschungsrelationen an

(37) $$\begin{cases} [Q_\alpha(\mathfrak{r}), Q_\beta(\mathfrak{r}')] = [\mathfrak{P}^\alpha(\mathfrak{r}), \mathfrak{P}^\beta(\mathfrak{r}')] = 0, \\ [\mathfrak{P}^\alpha(\mathfrak{r}), Q_\beta(\mathfrak{r}')] = \omega\, \delta_\beta^\alpha\, \delta(\mathfrak{r} - \mathfrak{r}'), \; \omega = \frac{hc}{2\pi i}, \end{cases}$$

sowie die Feldgleichungen [15]

(38) $$\begin{cases} [\overline{\mathfrak{H}}, Q_\alpha] = \omega \dot{Q}_\alpha \\ [\overline{\mathfrak{H}}, \mathfrak{P}^\alpha] = \omega \dot{\mathfrak{P}}^\alpha, \end{cases}$$

wobei die Bezeichnung

(39) $$\overline{\mathfrak{A}} \equiv \int \mathfrak{A}\, dx^1 dx^2 dx^3$$

gebraucht ist; das Integrationsgebiet muß so gewählt werden, daß die Feldgrößen am Rande *konstante* Werte annehmen und zwar solche, daß $\mathfrak{L}$ dort verschwindet. [16]

Zu (37) und (38) kommen noch als Nebenbedingungen die eigentlichen Identitäten (29) $\mathfrak{F}_r = 0$ hinzu. Aber es muß bewiesen werden, daß es erlaubt ist, die q-Zahlen $\mathfrak{F}_r$ alle gleichzeitig gleich Null zu setzen; mit anderen Worten, daß die $\mathfrak{F}_r$ untereinander vertauschbar sind, wenigstens auf Grund der Nebenbedingungen $\mathfrak{F}_r = 0$ selber.

Die jetzt folgenden Betrachtungen dienen nicht nur zu diesem Zwecke, sondern sind auch für den später zu erbringenden Kovarianzbeweis grundlegend.

Wir definieren zunächst den Impuls-Energie-Pseudotensor [1])

(40) $$\mathfrak{G}_\mu^\nu = \underline{\mathfrak{P}^{\alpha\nu} Q_{\alpha,\mu}} - \delta_\mu^\nu \mathfrak{L},$$

sodann die Impuls-Energie-Pseudodichte

(41) $$\mathfrak{G}_\mu \equiv \mathfrak{G}_\mu^4 = \underline{\mathfrak{P}^\alpha Q_{\alpha,\mu}} - \delta_\mu^4 \mathfrak{L},$$

deren vierte Pseudokomponente die Hamiltonfunktion (34) ist:

$$\mathfrak{H} \equiv \mathfrak{G}_4 \equiv \mathfrak{G}_4^4 .$$

Die Komponenten des Gesamtimpulses sind dann $\overline{\mathfrak{G}}_{\bar\nu}$, die Gesamtenergie $\overline{\mathfrak{H}}$.

Die V.-R. (Vertauschungsrelationen) von $\overline{\mathfrak{H}}$ mit den Q_α, $\mathfrak{P}^\alpha$ sind durch (38) gegeben. Was die $\overline{\mathfrak{G}}_{\bar\nu}$ betrifft, so finden wir zunächst auf Grund von (37)

(42) $$\begin{cases} [\mathfrak{G}_{\bar\nu}(\mathfrak{r}), Q_\alpha(\mathfrak{r}')] = \omega \dfrac{\partial Q_\alpha}{\partial x^{\bar\nu}} \delta(\mathfrak{r} - \mathfrak{r}'), \\[2ex] [\mathfrak{G}_{\bar\nu}(\mathfrak{r}), \mathfrak{P}^\alpha(\mathfrak{r}')] = -\omega \mathfrak{P}^\alpha \dfrac{\partial \delta(\mathfrak{r} - \mathfrak{r}')}{\partial x^{\bar\nu}}, \end{cases}$$

sodann

$$[\overline{\mathfrak{G}}_{\bar\nu}, \Phi(Q, \mathfrak{P})] = \omega \frac{d\Phi}{dx^{\bar\nu}},$$

allgemeiner also

(43) $$\omega \frac{d\Phi}{dx^\nu} = [\overline{\mathfrak{G}}_\nu, \Phi(Q, \mathfrak{P}, x)] + \omega \frac{\partial \Phi}{\partial x^\nu} .$$

Daraus folgt unmittelbar [17]

(44) $$[\overline{\mathfrak{G}}_\nu, \overline{\mathfrak{G}}_\mu] = 0 :$$

ein Ausdruck für die Vertauschbarkeit der Differentiationen $\dfrac{d}{dx^\nu}$, dessen physikalischer Inhalt in der zeitlichen Konstanz der $\overline{\mathfrak{G}}_\nu$ als Folge der Gleichungen (38), (37) besteht.[2])

---

1) Der Vorsatz „Pseudo" deutet an, daß die betreffenden Größen keine Tensoren sind.

2) Falls die $\lambda^r x^4$ explizite enthalten, gilt (44) erst auf Grund der Nebenbedingungen (29).

§ 5. Quantenmechanischer Ausdruck der infinitesimalen Transformation der Gruppe

In diesem Paragraphen beweisen wir den Satz: [18]

$$\omega\, \delta^* \Phi(Q, \mathfrak{P}) = [\overline{\mathfrak{M}, \Phi}], \tag{45}$$

wobei

$$\mathfrak{M} = \underline{\mathfrak{P}^\alpha\, \delta Q_\alpha} - \mathfrak{G}_\mu\, \delta x^\mu. \tag{46}$$

Dies soll auf Grund der Feldgleichungen (38) und der V.-R. (37) gelten, unter der Voraussetzung (13), daß $\mathfrak{L}$ eine skalare Dichte ist.

Um diesen Satz zu beweisen, genügt es zu zeigen, daß

$$\left\{\begin{aligned} \omega\, \delta^* Q_\alpha &= [\overline{\mathfrak{M}, Q_\alpha}], \\ \omega\, \delta^* \mathfrak{P}^\alpha &= [\overline{\mathfrak{M}, \mathfrak{P}^\alpha}]. \end{aligned}\right. \tag{47}$$

Nach (37) und (42) ist, wenn man bedenkt, daß δQ_α nach (2) nur die Q_α (nicht die $\mathfrak{P}^\alpha$) enthält,

$$[\overline{\mathfrak{M}, Q_\alpha}] = \omega\, \delta Q_\alpha - \frac{d\, Q_\alpha}{d\, x^{\bar\mu}}\, \delta\, x^{\bar\mu} - [\overline{\delta\, x^4 \cdot \mathfrak{H}, Q_\alpha}].$$

Nun ist, nach H. P. I, Formel (20),

$$\left\{\begin{aligned} [\overline{\delta\, x^4\, \mathfrak{H}, Q_\alpha}] &= \omega\, \frac{\partial\, (\delta\, x^4\, \mathfrak{H})}{\partial\, \mathfrak{P}^\alpha} = \delta\, x^4 \cdot [\overline{\mathfrak{H}, Q_\alpha}], \\ [\overline{\delta\, x^4\, \mathfrak{H}, \mathfrak{P}^\alpha}] &= -\, \omega \left\{ \frac{\partial\, (\delta\, x^4\, \mathfrak{H})}{\partial\, Q_\alpha} - \frac{d}{d\, x^{\bar\nu}}\, \frac{\partial\, (\delta\, x^4\, \mathfrak{H})}{\partial\, Q_{\alpha, \bar\nu}} \right\} \\ &= \delta\, x^4 [\overline{\mathfrak{H}, \mathfrak{P}^\alpha}] + \omega\, \frac{d\, \delta\, x^4}{d\, x^{\bar\nu}}\, \frac{\partial\, \mathfrak{H}}{\partial\, Q_{\alpha, \bar\nu}}. \end{aligned}\right. \tag{48}$$

Folglich gilt tatsächlich, mit Rücksicht auf (4), die erste Formel (47), wenn man noch die erste Feldgleichung (38) benutzt.

Analog findet man unter Berücksichtigung der zweiten Formel (48), der zweiten Feldgleichung (38) und der Formel (33) [19]

$$\left\{\begin{aligned} \frac{1}{\omega}\, [\overline{\mathfrak{M}, \mathfrak{P}^\alpha}] &= -\, \underline{\mathfrak{P}^\beta\, \frac{\partial\, \delta\, Q_\beta}{\partial\, Q_\alpha}} - \frac{d}{d x^{\bar\nu}}\, (\mathfrak{P}^\alpha\, \delta\, x^{\bar\nu}) \\ &\qquad - \frac{d\, \mathfrak{P}^\alpha}{d x^4}\, \delta x^4 + \mathfrak{P}^{\alpha \bar\nu}\, \frac{d\, \delta\, x^4}{d x^{\bar\nu}} \\ &= -\, \underline{\mathfrak{P}^\beta\, \frac{\partial\, \delta\, Q_\beta}{\partial\, Q_\alpha}} + \mathfrak{P}^{\alpha\, \nu}\, \frac{d\, \delta\, x^4}{d x^\nu} - \frac{d}{d x^\nu}\, (\mathfrak{P}^\alpha\, \delta\, x^\nu). \end{aligned}\right. \tag{49}$$

Es bleibt nur noch übrig zu zeigen, daß die rechte Seite von (49) gleich $\delta^* \mathfrak{P}^\alpha$ ist. Berechnen wir also direkt $\delta \mathfrak{P}^\alpha$, oder vielmehr allgemeiner $\delta \mathfrak{P}^{\alpha\nu}$. Zunächst gilt:

$$\delta \mathfrak{P}^{\alpha\nu} = \delta \left(\frac{\partial \mathfrak{L}}{\partial Q_{\alpha,\nu}} \right) = \frac{\partial (\delta \mathfrak{L})}{\partial Q_{\alpha,\nu}} - \underline{\frac{\partial \mathfrak{L}}{\partial Q_{\beta,\mu}} \frac{\partial \delta Q_{\beta,\mu}}{\partial Q_{\alpha,\nu}}}, \tag{50}$$ [20]

und zwar bei c-Zahlen allgemein, bei q-Zahlen jedenfalls, wenn $\mathfrak{L}$ die Form (1) hat und $\frac{\partial \delta Q_{\beta,\mu}}{\partial Q_{\alpha,\nu}}$ die $\dot{Q}_\alpha$ bzw. $\mathfrak{P}^\alpha$ nicht enthält. Daß letzteres in unserem Falle zutrifft, zeigt die Formel (6), welche ergibt:

$$\frac{\partial}{\partial Q_{\alpha,\nu}} \delta Q_{\beta,\mu} = \frac{\partial}{\partial Q_{\alpha,\nu}} \left\{ \frac{d}{dx^\mu} \delta Q_\beta - Q_{\beta,\varrho} \frac{d \delta x^\varrho}{dx^\mu} \right\}$$
$$= \frac{\partial \delta Q_\beta}{\partial Q_\alpha} \delta^\nu_\mu - \frac{d \delta x^\nu}{dx^\mu} \delta^\alpha_\beta .$$

Dies, in (50) eingesetzt, liefert

$$\delta \mathfrak{P}^{\alpha\nu} = - \underline{\mathfrak{P}^{\beta\nu} \frac{\partial \delta Q_\beta}{\partial Q_\alpha}} + \mathfrak{P}^{\alpha\mu} \frac{d \delta x^\nu}{dx^\mu} + \frac{\partial (\delta \mathfrak{L})}{\partial Q_{\alpha,\nu}} ;$$

benutzen wir jetzt (13), so kommt

$$\delta \mathfrak{P}^{\alpha\nu} = - \underline{\mathfrak{P}^{\beta\nu} \frac{\partial \delta Q_\beta}{\partial Q_\alpha}} + \mathfrak{P}^{\alpha\mu} \frac{d \delta x^\nu}{dx^\mu} - \mathfrak{P}^{\alpha\nu} \frac{d \delta x^\mu}{dx^\mu} ; \tag{51}$$

d. h. wie der Vergleich mit (11) lehrt: $\mathfrak{P}^{\alpha\nu}$ ist eine Tensordichte. Aus (51) folgt nun sofort mit Rücksicht auf (4) für $\delta^* \mathfrak{P}^\alpha \equiv \delta^* \mathfrak{P}^{\alpha 4}$ der Ausdruck (49).

Somit ist die Formel (45) bewiesen.

§ 6. Die $\overline{\mathfrak{F}_r}$ als spezielle infinitesimale Transformationen

Betrachten wir einen bestimmten, aber beliebigen Schnitt $x^4 = x_0^4$. Auf diesem Schnitt betrachten wir die Transformationen unserer Gruppe (2), welche durch die Forderungen

$$\left\{ \begin{aligned} & (\xi^r)_{x^4 = x_0^4} = \left(\frac{\partial \xi^r}{\partial x^\sigma} \right)_{x^4 = x_0^4} = \cdots = \left(\frac{\partial^{j-1} \xi^r}{\partial x^\sigma \cdots \partial x^\tau} \right)_{x^4 = x_0^4} = 0 , \\ & \left(\frac{\partial^j \xi^r}{\partial x^\sigma \cdots \partial x^\tau} \right)_{x^4 = x_0^4} = 0, \text{ wenn nicht alle } \sigma, \ldots, \tau \text{ gleich 4 sind,} \\ & \left[\frac{\partial^j \xi^r}{(\partial x^4)^j} \right]_{x^4 = x_0^4} = \varepsilon^r \end{aligned} \right. \tag{52}$$

definiert sind, wobei die ε^r beliebige Raumfunktionen sind.

Wegen der Voraussetzung (3) führen diese Transformationen nicht aus dem Schnitt $x^4 = x_0{}^4$ heraus. Sie bilden in jedem Punkte dieses Schnittes eine endliche kontinuierliche Untergruppe der Gruppe (2), deren infinitesimale Transformation nach (45) und (46) gegeben ist durch

$$\omega\, \delta' \Phi(Q, \mathfrak{P}) = [\varepsilon^r \mathfrak{F}_r, \Phi].$$ [21]

(Hierin sind Q, $\mathfrak{P}$, $\mathfrak{F}_r$ für $x^4 = x_0{}^4$ zu nehmen.)

Der zweite Fundamentalsatz von Lie über endliche Transformationsgruppen besagt, angewandt auf diese Untergruppe, daß in jedem Punkte des Schnittes

$$[\mathfrak{F}_r, [\mathfrak{F}_s, \Phi]] - [\mathfrak{F}_s, [\mathfrak{F}_r, \Phi]] = c^t_{rs} [\mathfrak{F}_t, \Phi]$$

gilt, wo die c^t_{rs} die vom Punkte $(x^1, x^2, x^3, x_0{}^4)$ abhängigen „Strukturkonstanten" der Gruppe sind. Nach der Jacobischen Identität über Klammersymbole wird die linke Seite einfach gleich

$$[[\mathfrak{F}_s, \mathfrak{F}_r], \Phi];$$

also bekommen wir [22]

$$[\mathfrak{F}_s, \mathfrak{F}_r] = c^t_{rs} \mathfrak{F}_t. \tag{53}$$

Daraus folgt die zur Begründung des in § 4 dargelegten Verfahrens noch nötige Tatsache, daß *auf Grund von* $\mathfrak{F}_r = 0$ die $\mathfrak{F}_r$ untereinander vertauschbar sind.

§ 7. Die infinitesimale Transformation $\mathfrak{M}$ als Integral der Bewegung

Kehren wir einen Augenblick zur reinen c-Zahltheorie zurück. Setzen wir

$$\mathfrak{M}^\nu = \underline{\mathfrak{P}^{\alpha\nu}\, \delta Q_\alpha} - \mathfrak{S}^\nu_\mu\, \delta x^\mu \tag{54}$$

und

$$\mathfrak{L}^\alpha = \frac{\partial \mathfrak{L}}{\partial Q_\alpha} - \frac{d}{dx^\nu} \frac{\partial \mathfrak{L}}{\partial Q_{\alpha,\nu}}, \tag{55}$$

so ist, wie leicht zu sehen, die Voraussetzung (13) gleichbedeutend mit [23]

$$\frac{d\mathfrak{M}^\nu}{dx^\nu} + \mathfrak{L}^\alpha\, \delta^* Q_\alpha = 0; \tag{56c}$$

berücksichtigt man nun, daß nach (46) und (54)

$$\mathfrak{M} \equiv \mathfrak{M}^4$$

und entsprechend der Bedeutung der Bezeichnung (39) [24]

$$\overline{\frac{d\,\mathfrak{M}^{\bar{\nu}}}{d x^{\bar{\nu}}}} = 0$$

ist, so folgt aus (56c)

(57c) $$\frac{d\,\overline{\mathfrak{M}}}{d x^4} = - \overline{\mathfrak{L}^\alpha\, \partial^* Q_\alpha}\,.$$

Nun sind bekanntlich die Hamiltonschen Gleichungen (38) [vermöge der eigentlichen Identitäten (29)] äquivalent mit den Lagrangegleichungen

$$\mathfrak{L}^\alpha = 0\,.$$

Nach (57c) gilt somit, auf Grund von (13) und (38)

(58) $$\frac{d\,\overline{\mathfrak{M}}}{d x^4} = 0\,.$$

Die Gleichung (56c) läßt sich nicht auf q-Zahlen übertragen. Die Ableitung von (58) gelingt jedoch unter Benutzung der nämlichen Voraussetzungen (13) und (38), bloß in etwas anderem Zusammenhang. Die beiden Relationen (13) und (38) wurden zur Ableitung der Formeln (43) und (45) wesentlich gebraucht. Wenden wir diese letzteren auf die Identität (5) an, wobei Φ nur von Q und $\mathfrak{P}$ abhängen möge: [25]

$$\left[\overline{\mathfrak{M}}, [\overline{\mathfrak{G}}_\nu, \Phi]\right] = \left[\overline{\mathfrak{G}}_\nu, [\overline{\mathfrak{M}}, \Phi]\right] + \left[\omega\, \frac{\partial\, \overline{\mathfrak{M}}}{\partial x^\nu}, \Phi\right],$$

$$\left[[\overline{\mathfrak{G}}_\nu, \overline{\mathfrak{M}}] + \omega\, \frac{\partial\, \overline{\mathfrak{M}}}{\partial x^\nu}, \Phi\right] = 0$$

nach der Jacobischen Identität, oder schließlich

$$\left[\frac{d\,\overline{\mathfrak{M}}}{d x^\nu}, \Phi\right] = 0.$$

Insbesondere ist also $\frac{d\,\overline{\mathfrak{M}}}{d x^4}$ eine (eventuell von x^4 abhängige) c-Zahl. Nun kann diese c-Zahl, als Summe von lauter q-Zahlen, nichts anderes als Null sein. Diesen Schluß bestätigt übrigens die etwas mühsame Ausrechnung von $\frac{d\,\overline{\mathfrak{M}}}{d x^4}$.

Aus (58) lassen sich interessante Schlüsse ziehen über den Zusammenhang von $\overline{\mathfrak{M}}$ mit den Funktionen $\mathfrak{F}_r$. Durch partielle Integrationen wird $\overline{\mathfrak{M}}$ in die Form:

$$\overline{\mathfrak{M}} = \int dx^1\, dx^2\, dx^3 \sum_{i=0}^{i=j} \mathfrak{N}_r{}^i \frac{\partial^i \xi^r}{(\partial x^4)^i} \tag{59}$$

gebracht, wobei

$$\mathfrak{N}_r^j \equiv \mathfrak{F}_r. \tag{60}$$

(58) drückt sich dann folgendermaßen aus:

$$\int dx^1\, dx^2\, dx^3 \sum_{i=0}^{i=j} \mathfrak{N}_r{}^i \frac{\partial^{i+1} \xi^r}{(\partial x^4)^{i+1}}$$
$$= - \int dx^1\, dx^2\, dx^3 \sum_{i=0}^{i=j} \frac{d\,\mathfrak{N}_r{}^i}{dx^4} \frac{\partial^i \xi^r}{(\partial x^4)^i}\,.$$

Daraus folgert man durch Koeffizientenvergleich

$$\mathfrak{N}_r{}^i = - \frac{d\,\mathfrak{N}_r^{i+1}}{dx^4} \qquad (i = 0, 1, \ldots, j-1) \tag{61}$$

und

$$\mathfrak{N}_r^j = 0, \quad \frac{d\,\mathfrak{N}_r{}^0}{dx^4} = 0\,. \tag{62}$$

Aus (60) und (61) ergibt sich

$$\mathfrak{N}_r{}^i = (-1)^{j-i} \frac{d^{j-i}\mathfrak{F}_r}{(dx^4)^{j-i}} \qquad (i = 0, 1, \ldots, j), \tag{63}$$

also für $\overline{\mathfrak{M}}$ die merkwürdige Gestalt

$$\overline{\mathfrak{M}} = \int dx^1\, dx^2\, dx^3 \sum_{i=0}^{i=j} (-1)^{j-i} \frac{d^{j-i}\mathfrak{F}_r}{(dx^4)^{j-i}} \frac{\partial^i \xi^r}{(\partial x^4)^i}\,. \tag{63'}$$

Die erste Identität (62) ist nach (60) trivialerweise $\mathfrak{F}_r = 0$, die zweite aber besagt, daß auf Grund der Feldgleichungen und der Identitäten (29)

$$\frac{d^{j+1}\mathfrak{F}_r}{(dx^4)^{j+1}} = 0\,. \tag{64}$$

Dies liefert die Antwort auf die Frage, inwieweit man durch abermalige Differentiation der Nebenbedingungen (29) neue Relationen bekommt. [26]

Ist insbesondere $j = 1$, so sind die einzigen neuen Gleichungen $\frac{d\mathfrak{F}_r}{dx^4} = 0$, d. h.

$$[\overline{\mathfrak{H}}, \mathfrak{F}_r] + \omega \frac{\partial \mathfrak{F}_r}{\partial x^4} = 0 .$$

Ist nun die Lagrangefunktion von der Gestalt (1), d. h. gilt (35), so wird die letzte Gleichung mit Rücksicht auf (53)

$$[\overline{\mathfrak{H}}_0, \mathfrak{F}_r] + \omega \frac{\partial \mathfrak{F}_r}{\partial x^4} + c^t_{rs} \lambda^s \mathfrak{F}_t = 0$$

oder schließlich vermöge $\mathfrak{F}_r = 0$

$$[\overline{\mathfrak{H}}_0, \mathfrak{F}_r] + \omega \frac{\partial \mathfrak{F}_r}{\partial x^4} = 0 :$$

da weder die Nebenbedingungen noch diese neuen Gleichungen die λ^r enthalten, so bleiben dieselben wesentlich unbestimmt. (Anders aber, falls $j > 1$, denn bereits $\frac{d^2 \mathfrak{F}_r}{(dx^4)^2} = 0$ enthält die λ^r). [27]
Infolge der wesentlichen Unbestimmtheit der λ^r fehlen r_0 Feldgleichungen von der Form [28]

$$\omega \dot{\mathfrak{P}}^a = [\overline{\mathfrak{H}}, \mathfrak{P}^a] ;$$

zum Ersatz reichen gerade die Gleichungen

$$\mathfrak{R}_r{}^0 \equiv \frac{d\mathfrak{F}_r}{dx^4} = 0 , \quad \text{d. h.} \quad [\overline{\mathfrak{H}}_0, \mathfrak{F}_r] + \omega \frac{\partial \mathfrak{F}_r}{\partial x^4} = 0$$

aus. [29]

Im Falle $j = 0$ werden die fehlenden Feldgleichungen ersetzt durch die Identitäten $\mathfrak{F}_r = 0$ selber, die sich gemäß (64), d. h. $\frac{d\mathfrak{F}_r}{dx^4} = 0$ auf Grund der Feldgleichungen mit der Zeit fortpflanzen.

Eine letzte Bemerkung wollen wir noch an die Formel (63′) anknüpfen. Fragen wir nach der Untergruppe unserer Gruppe, die alle Punkte des Schnittes $x^4 = x_0{}^4$ invariant läßt: diese Untergruppe ist offenbar ein *Normalteiler*. Die Bedingungen

$$\delta x^\nu = 0 \quad \text{für} \quad x^4 = x_0{}^4$$

bedeuten

$$(\xi^r)_{x^4 = x_0^4} = \left(\frac{\partial \xi^r}{\partial x^\sigma}\right)_{x^4 = x_0^4} = \cdots = \left[\frac{\partial^k \xi^r}{\partial x^\sigma \cdots \partial x^\tau}\right]_{x^4 = x_0^4} = 0 ;$$

die Infinitesimaltransformation lautet demnach gemäß (63′)

$$\overline{\mathfrak{S}} = \int dx^1\, dx^2\, dx^3 \sum_{i=k+1}^{i=j} (-1)^{j-i} \frac{d^{j-i}\mathfrak{F}_r}{(dx^4)^{j-i}}\, \varepsilon_i^r, \tag{65}$$

wobei $\frac{d^{j-i}\mathfrak{F}_r}{(dx^4)^{j-i}}$ für $x^4 = x_0^4$ zu nehmen ist und die

$$\varepsilon_i^r \equiv \left[\frac{\partial^i \xi^r}{(\partial x^4)^i}\right]_{x^4 = x_0^4}$$

willkürliche Raumfunktionen sind. Die Gruppe $\overline{\mathfrak{S}}$ ist in jedem Punkte des Schnittes eine $r_0(j-k)$-parametrige invariante Untergruppe. Die im § 6 betrachtete Gruppe (52) ist eine Untergruppe davon. [30]

§ 8. Kovarianz des Verfahrens gegenüber der Gruppe

Mittels der im Vorangehenden gewonnenen Ergebnisse sind wir nunmehr imstande, die Frage nach der Kovarianz des Verfahrens leicht zu erledigen.

Die Formel (45) besagt, daß bei einer beliebigen Transformation der Gruppe jedes Funktional $\Phi(Q, \mathfrak{P})$ einer unitären Ähnlichkeitstransformation von der Gestalt

$$\Phi' = S^{-1}\Phi S \tag{66}$$

unterworfen ist, wobei nach (58) S zeitunabhängig ist.

Ferner gilt, wie leicht einzusehen[1]), Formel (45) auch für infinitesimale Transformationen $\overline{\mathfrak{N}}$ der Gruppe, d. h. es ist,

---

1) Ersetzt man in

$$\Phi' = \Phi + \frac{1}{\omega}[\overline{\mathfrak{N}}, \Phi]$$

Φ durch

$$\tilde{\Phi} = \Phi + \frac{1}{\omega}[\overline{\mathfrak{M}}, \Phi]$$

und Φ' durch

$$\tilde{\Phi}' = \Phi' + \frac{1}{\omega}[\overline{\mathfrak{M}}, \Phi'],$$

so kommt nach leichter Rechnung

$$\tilde{\Phi}' = \tilde{\Phi} + \frac{1}{\omega}\left[\overline{\mathfrak{N}} + \frac{1}{\omega}[\overline{\mathfrak{M}}, \overline{\mathfrak{N}}],\ \tilde{\Phi}\right].$$

Vgl. auch E. Noether, Gött. Nachr. 1918, S. 252. [31]

wenn man alle Feldgrößen der infinitesimalen Transformation $\overline{\mathfrak{M}}$ unterwirft,

(67) $$\omega\, \delta^* \overline{\mathfrak{N}} = [\overline{\mathfrak{M}}, \overline{\mathfrak{N}}],$$

daher allgemeiner

(67′) $$\overline{\mathfrak{N}}' = S^{-1}\, \overline{\mathfrak{N}}\, S\,.$$

Aus (66) folgt unmittelbar die Invarianz der kanonischen V.-R. (37). Nach (35) besteht die Hamiltonfunktion aus einem nur von Q und $\mathfrak{P}$ abhängigen Funktional $\overline{\mathfrak{H}_0}$ und einem Anteil $\overline{\lambda^r \mathfrak{F}_r}$, der nach § 6 eine spezielle infinitesimale Transformation $\overline{\mathfrak{N}}$ darstellt. Wegen (66) und (67′) erleiden also auch die kanonischen Feldgleichungen (38) eine (zeitlich konstante) unitäre Transformation, bei der sie bekanntlich invariant bleiben.

Es bleibt noch übrig, die Variation der linken Seiten $\mathfrak{F}_r$ der Identitäten (29) zu untersuchen. Nach (67) beträgt sie

(68) $$\omega\, \delta^* \mathfrak{F}_r = [\overline{\mathfrak{M}}, \mathfrak{F}_r]\,.$$

Nun gilt infolgedessen, daß die durch (65) definierte Gruppe $\overline{\mathfrak{S}}$ eine invariante Untergruppe ist, [32]

(68′) $$[\overline{\mathfrak{M}}, \mathfrak{F}_r] = \sum_{i=k+1}^{i=j} \alpha_i^{rs}\, \frac{d^{j-i}\mathfrak{F}_s}{(dx^4)^{j-i}}\,.$$

Gemäß (68) und (68′) sind also die $\delta^* \mathfrak{F}_r = 0$, d. h. die eigentlichen Identitäten $\mathfrak{F}_r = 0$ invariant, und zwar vermöge der Identitäten selber und eventuell deren zeitlichen Ableitungen.

§ 9. Erweiterung der Theorie auf den „zweiten Fall“ des § 1

Wir deuten kurz an, wie sich die vorige Theorie auf den am Ende des § 1 definierten „zweiten Fall“ ausdehnt.

Unsere Gruppe habe also die einfache Form:

(69) $$\begin{cases} \delta x^\nu = 0 \\ \delta Q_\alpha = c_{\alpha r}\, \xi^r\,. \end{cases}$$

Mit

(14) $$\mathfrak{L}' \equiv \frac{d}{dx^\nu}\left[\underline{\mathfrak{f}^{\nu,\alpha\varrho}(Q)\, Q_{\alpha.\varrho}}\right]$$

ist nach (12)

(70) $$\delta(\mathfrak{L} + \mathfrak{L}') = 0\,.$$

1. Berechnen wir zunächst $\delta \mathfrak{L}'$. Ich behaupte, daß $\delta \mathfrak{L}'$ die Form

$$(71) \qquad \delta \mathfrak{L}' = \frac{d}{d x^\nu} (\mathfrak{R}^{\alpha \nu} \, \delta Q_\alpha)$$

oder

$$(72) \qquad \delta \mathfrak{L}' = \frac{d}{d x^\nu} (\mathfrak{I}_r^{\ \nu} \, \xi^r)$$

annimmt.

Denn wir bekommen zunächst

$$\delta \mathfrak{L}' = \frac{d}{d x^\nu} \left\{ \frac{\partial \mathfrak{f}^{\nu, \alpha \varrho}}{\partial Q_\beta} c_{\beta r} \, \xi^r \, Q_{\alpha, \varrho} + \mathfrak{f}^{\nu, \alpha \varrho} \frac{d (c_{\alpha r} \, \xi^r)}{d x^\varrho} \right\};$$

setzen wir

$$(73) \qquad \mathfrak{r}^{\alpha \nu} = - \frac{d \mathfrak{f}^{\nu, \alpha \varrho}}{d x^\varrho} + Q_{\beta, \varrho} \frac{\partial \mathfrak{f}^{\nu, \beta \varrho}}{\partial Q_\alpha}$$

und

$$(74) \qquad \mathfrak{I}_r^{\ \nu} = \mathfrak{r}^{\alpha \nu} c_{\alpha r},$$

so wird

$$(75) \qquad \delta \mathfrak{L}' = \frac{d}{d x^\nu} \left\{ \mathfrak{I}_r^{\ \nu} \, \xi^r + \frac{d}{d x^\varrho} (\mathfrak{f}^{\nu, \alpha \varrho} \, c_{\alpha r} \, \xi^r) \right\}.$$

Jetzt benutzen wir (70) und drücken aus, daß die Koeffizienten der zweiten Ableitungen der ξ^r identisch verschwinden. Da $\mathfrak{L}$ keine zweiten Ableitungen der ξ^r enthält, so wird nach (75)

$$(76) \qquad (\mathfrak{f}^{\nu, \alpha \varrho} + \mathfrak{f}^{\varrho, \alpha \nu}) \, c_{\alpha r} \equiv 0 .$$

Infolgedessen reduziert sich (75) zu

$$\delta \mathfrak{L}' = \frac{d}{d x^\nu} (\mathfrak{I}_r^{\ \nu} \, \xi^r).$$

Setzen wir noch

$$(77) \qquad \mathfrak{R}^{\alpha \nu} = \mathfrak{r}^{\alpha \nu}$$

und bemerken, daß statt (74) auch

$$(74) \qquad \mathfrak{I}_r^{\ \nu} = \mathfrak{R}^{\alpha \nu} c_{\alpha r}$$

geschrieben werden darf, so haben wir die Formeln (71) und (72) bewiesen.

2. Jetzt stellen wir die Analoga der Identitäten (28) auf, die im ersten Falle auch eigentliche Identitäten (29) enthielten.

Dazu haben wir bloß die Koeffizienten der $\frac{d\xi^r}{dx^\nu}$ in (70) gleich Null zu setzen. Das liefert uns

$$(78)\qquad \underline{(\mathfrak{P}^{\alpha\nu} + \mathfrak{N}^{\alpha\nu})\; c_{\alpha r}} = 0\,,$$

Insbesondere für $\nu = 4$:

$$\underline{(\mathfrak{P}^{\alpha} + \mathfrak{N}^{\alpha 4})\,c_{\alpha r}} = 0\,,$$

oder, indem wir wiederum

$$\mathfrak{F}_r = \underline{\mathfrak{P}^{\alpha}\,c_{\alpha r}} \quad \text{und} \quad \mathfrak{J}_r^{\,4} = \mathfrak{J}_r$$

setzen,

$$(79)\qquad \mathfrak{F}_r + \mathfrak{J}_r = 0\,.$$

3. Die Identitäten (79) sind eigentliche, d. h. es ist

$$(80)\qquad \frac{\partial\,\mathfrak{J}_r}{\partial\,\dot{Q}_\alpha} = 0\,.$$

Allgemeiner wollen wir statt (80) beweisen, daß

$$(80')\qquad \underline{\frac{\partial\,(\mathfrak{r}^{\beta\varrho}\,c_{\beta r})}{\partial\,Q_{\alpha,\nu}} = -\,\frac{\partial\,(\mathfrak{r}^{\beta\nu}\,c_{\beta r})}{\partial\,Q_{\alpha,\varrho}}}\,,$$

woraus (80) nach (74) für $\nu = \varrho = 4$ folgt.

Zu dem Zweck setzen wir den Koeffizienten von ξ_r in (70) gleich Null: es ist ein in den zweiten Ableitungen $Q_{\alpha,\varrho\nu}$ linearer Ausdruck, mit nur von Q abhängigen Koeffizienten. Da dieser Ausdruck für beliebige $Q_{\alpha,\varrho\nu}$ identisch verschwindet, können wir insbesondere den $Q_{\alpha,\varrho\nu}$ c-Zahlwerte zuschreiben und dann die Koeffizienten der $Q_{\alpha,\varrho\nu}$ getrennt gleich Null setzen. Unter Benutzung der für beliebiges $\mathfrak{K}^\varrho(Q_\alpha;\, Q_{\alpha,\nu})$ gültigen Formel:

$$(81)\qquad \frac{\partial}{\partial\,Q_{\alpha,\nu}}\;\frac{d}{d x^\varrho}\,\mathfrak{K}^\varrho(Q_\alpha;\,Q_{\alpha,\nu}) = \frac{\partial\,\mathfrak{K}^\nu}{\partial\,Q_\alpha} + \frac{d}{d x^\varrho}\;\frac{\partial\,\mathfrak{K}^\varrho}{\partial\,Q_{\alpha,\nu}}\,,$$

finden wir für diese Koeffizienten nach (71) und (73) [33]

$$\underline{\frac{\partial\,(\mathfrak{r}^{\beta\varrho}\,c_{\beta r})}{\partial\,Q_{\alpha,\nu}} + \frac{\partial\,(\mathfrak{r}^{\beta\nu}\,c_{\beta r})}{\partial\,Q_{\alpha,\varrho}}}\,,$$

deren Nullsetzen (80′) ergibt.

Gemäß (81) folgt übrigens aus (73) [34]

$$(82)\qquad \frac{\partial\,\mathfrak{r}^{\alpha\nu}}{\partial\,Q_{\beta,\varrho}} = \frac{\partial\,\mathfrak{f}^{\nu,\beta\varrho}}{\partial\,Q_\alpha} - \frac{\partial\,\mathfrak{f}^{\nu,\alpha\varrho}}{\partial\,Q_\beta} = \frac{\partial\,\mathfrak{N}^{\alpha\nu}}{\partial\,Q_{\beta,\varrho}}\,;$$

danach können wir statt (80') auch

$$\frac{\partial(\mathfrak{R}^{\beta\varrho}c_{\beta r})}{\partial Q_{\alpha,\nu}} = -\frac{\partial(\mathfrak{R}^{\beta\nu}c_{\beta r})}{\partial Q_{\alpha,\varrho}} \tag{83}$$

schreiben.

4. Die Ausführungen der §§ 3 und 4 sind auf den jetzigen Fall wörtlich übertragbar, *indem* $\mathfrak{P}^{\alpha} + \mathfrak{R}^{\alpha 4}$ *die Rolle von* $\mathfrak{P}^{\alpha}$ *vertritt.*

Der im § 5 abgeleitete Ausdruck $\overline{\mathfrak{M}}$ der infinitesimalen Transformation erleidet eine analoge Modifikation, da $\mathfrak{P}^{\alpha\nu}$ jetzt keine Tensordichte mehr ist.[1])

Vielmehr ist jetzt nach (50) und (70)

$$\delta\mathfrak{P}^{\alpha\nu} = -\mathfrak{P}^{\beta\nu}\frac{\partial\,\delta Q_{\beta}}{\partial Q_{\alpha}} - \frac{\partial(\delta\mathfrak{L}')}{\partial Q_{\alpha,\nu}};$$

nach (71) ist aber, gemäß (81),

$$\frac{\partial(\delta\mathfrak{L}')}{\partial Q_{\alpha,\nu}} = \frac{\partial(\mathfrak{R}^{\beta\nu}\,\delta Q_{\beta})}{\partial Q_{\alpha}} + \frac{d}{d x^{\varrho}}\,\frac{\partial(\mathfrak{R}^{\beta\varrho}\,\delta Q_{\beta})}{\partial Q_{\alpha,\nu}},$$

d. h. mit Rücksicht auf (83)

$$\delta\mathfrak{P}^{\alpha\nu} = -\mathfrak{P}^{\beta\nu}\frac{\partial\,\delta Q_{\beta}}{\partial Q_{\alpha}} - \frac{\partial(\mathfrak{R}^{\beta\nu}\,\delta Q_{\beta})}{\partial Q_{\alpha}} + \frac{d}{d x^{\varrho}}\,\frac{\partial(\mathfrak{R}^{\beta\nu}\,\delta Q_{\beta})}{\partial Q_{\alpha,\varrho}}. \tag{84}$$

Insbesondere wird das wegen (80) für $\nu = 4$ [35]

$$\delta\mathfrak{P}^{\alpha} = -\mathfrak{P}^{\beta}\frac{\partial\,\delta Q_{\beta}}{\partial Q_{\alpha}} - \frac{\partial(\mathfrak{R}^{\beta 4}\,\delta Q_{\beta})}{\partial Q_{\alpha}} + \frac{d}{d x^{\overline{\varrho}}}\,\frac{\partial(\mathfrak{R}^{\beta 4}\,\delta Q_{\beta})}{\partial Q_{\alpha,\overline{\varrho}}},$$

d. h.

$$\omega\,\delta\mathfrak{P}^{\alpha} = [\overline{\mathfrak{N}}, \mathfrak{P}^{\alpha}], \tag{85}$$

mit

$$\mathfrak{N} = (\mathfrak{P}^{\alpha} + \mathfrak{R}^{\alpha 4})\,\delta Q_{\alpha}. \tag{86}$$

Wiederum wegen (80) ist auch [36]

$$\omega\,\delta Q_{\alpha} = [\overline{\mathfrak{N}}, Q_{\alpha}], \tag{87}$$

so daß wir in $\overline{\mathfrak{N}}$ die gesuchte Erweiterung von $\overline{\mathfrak{M}}$ haben.

---

1) Obwohl weder $\mathfrak{P}^{\alpha\nu}$ noch $\mathfrak{R}^{\alpha\nu}$ Tensordichten sind, läßt sich leicht zeigen, daß $\mathfrak{P}^{\alpha\nu} + \mathfrak{R}^{\alpha\nu}$ dennoch eine Tensordichte ist.

Aus dem Ausdruck (86) folgt genau wie im § 6, daß die linken Seiten $\mathfrak{F}_r + \mathfrak{J}_r$ der eigentlichen Identitäten auf Grund derselben untereinander vertauschbar sind.

Die Überlegungen von § 7 über die zeitliche Konstanz von $\overline{\mathfrak{M}}$, sowie der Kovarianzbeweis von § 8, lassen sich ohne weiteres auf $\overline{\mathfrak{N}}$ übertragen. Insbesondere spielen hier, da $j = 0$ vorausgesetzt wurde, die Identitäten $\mathfrak{F}_r + \mathfrak{J}_r = 0$ die Rolle der fehlenden Feldgleichungen.

§ 10. Bemerkung über die gleichzeitige Behandlung mehrerer Gruppen

Auf den Fall, daß die Lagrangefunktion mehrere Gruppen gestattet, ist die obige Theorie ohne weiteres anwendbar, wenn man bedenkt, daß die infinitesimale Transformation des direkten Produktes aller betrachteten Gruppen sich additiv aus denen der einzelnen Gruppen zusammensetzt. Insbesondere sind die sich auf jede einzelne Gruppe beziehenden $\mathfrak{F}_r$ nicht nur untereinander (vermöge $\mathfrak{F}_r = 0$) vertauschbar, sondern auch mit den zu den anderen Einzelgruppen gehörigen $\mathfrak{F}_r$. Es ist auch zulässig, daß der „erste Fall“ ($\mathfrak{L}$ ist eine Dichte) für gewisse Einzelgruppen, der im § 9 behandelte „zweite Fall“ dagegen für gewisse anderen auftritt. Dann sind für die letzteren die $\mathfrak{F}_r$ einfach durch $\mathfrak{F}_r + \mathfrak{J}_r$ zu ersetzen: sie sind wiederum nicht nur untereinander, sondern auch mit den anderen $\mathfrak{F}_r$ vertauschbar.

Diese Bemerkung hat zur Folge, daß man die einzelnen Gruppen, welche eine gegebene Lagrangefunktion gestattet, getrennt behandeln kann.

Zweiter Teil: **Anwendungen**

§ 11. Die Lagrangefunktion

Wir wollen eine Lagrangefunktion aufstellen, die sowohl das elektromagnetische und Materiefeld als auch das Gravitationsfeld umfaßt. Was das letztere betrifft, so übernehmen wir die von Fock[1]) und Weyl[2]) vorgeschlagene Theorie des Einkörperproblems: das Gravitationsfeld beschreiben wir durch

1) V. Fock, Ztschr. f. Phys. **57**. S. 261. 1929.

2) H. Weyl, Ztschr. f. Phys. **56**. S. 330. 1929.

Angabe in jedem Punkt von vier orthogonalen Vektoren $h_{i,\nu}$ $(i = 1, 2, 3, 4)$ und wir fordern, daß die Naturgesetze kovariant sind gegenüber einer *vom Punkte abhängigen* Lorentztransformation der „Vierbeine“ $h_{i,\nu}$; diese Kovarianz, die wir [37]
nach Levi-Cività[1]) „echte Beinkovarianz“ nennen, unterscheidet sich wesentlich von der in der Einsteinschen Theorie des Fernparallelismus geforderten „lokalen Beinkovarianz“, nach der alle Vierbeine miteinander starr verbunden sind (*konstante* Lorentztransformation der Vierbeine). Im Einklang mit Fock (und im Gegensatz zu Weyl) beschreiben wir das Materiefeld durch *vier*komponentige Wellenfunktionen $\psi \equiv (\psi_1, \psi_2, \psi_3, \psi_4)$. Für das elektromagnetische Feld wählen wir als Variable die Komponenten φ_μ des Viererpotentials.[2])

Die Lagrangefunktion setzt sich additiv zusammen aus drei Anteilen, die den drei genannten Feldern entsprechen (und gleichzeitig die Wechselwirkungen der Felder aufeinander enthalten).

Wenn

$$E_{\mu\nu} = \frac{\partial \varphi_\nu}{\partial x^\mu} - \frac{\partial \varphi_\mu}{\partial x^\nu} \tag{88}$$

den elektromagnetischen Feldtensor darstellt, so ist der Strahlungsanteil der Lagrangefunktion [39]

$$\mathfrak{E} = \tfrac{1}{4} E_{\mu\nu} \mathfrak{E}^{\mu\nu}; \tag{89}$$

dabei bedeutet

$$\mathfrak{E}^{\mu\nu} = E^{\mu\nu} h'$$

wo h' die Determinante der $h_{i,\nu}$ und $E^{\mu\nu}$ die kontravarianten Komponenten des Tensors $E_{\mu\nu}$ bezeichnet.

Um den Materieanteil aufzuschreiben, legen wir ein spezielles System Diracscher Matrizen fest.[3]) Gehen wir aus von den Paulischen Matrizen

$$\varrho_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \varrho_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \varrho_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{90}$$

---

1) Berliner Berichte 1929, S. 137.

2) Da wir $x^4 = ct$ gesetzt haben, ist $\varphi_4 = -\varphi$, wo φ das skalare Potential darstellt. [38]

3) Dasselbe weicht von dem Fockschen (a. a. O.) nur unwesentlich ab. Der wesentliche Zug der Spezialisierung ist $\alpha_4 = 1$ (in der Fockschen Bezeichnung $\alpha_0 = 1$).

so setzen wir

$$(91) \qquad \begin{cases} \alpha_{\bar{l}} = -i \begin{pmatrix} \varrho_{\bar{l}} & 0 \\ 0 & -\varrho_{\bar{l}} \end{pmatrix} \quad (\bar{l} = 1, 2, 3) \\ \alpha_4 = 1. \end{cases}$$

Führen wir noch die Bezeichnung

$$(92) \qquad e_{\bar{k}} = -1, \quad e_4 = 1$$

ein, so sind die Matrizen α_l hermitisch und haben die Vertauschungseigenschaft

$$(93) \qquad \alpha_m \alpha_k e_k + \alpha_k \alpha_m e_m = 2 e_m \delta_{mk}.$$

Ferner brauchen wir noch die Matrix

$$(94) \qquad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(worin die Einser zweireihige Einheitsmatrizen darstellen). [40]

In bezug auf die lateinischen Indizes ist über zweimal auftretende Indizes zu summieren, wobei die Faktoren e_k zur Abzählung der Indizes unberücksichtigt bleiben sollen. Neben die $h_{i,\nu}$ treten die kontravarianten h_i^ν und es gelten die Relationen [41]

$$(95) \qquad \begin{cases} h_k^\nu h_{l,\nu} = e_k \delta_{kl}, \\ e_k h_k^\nu h_{k,\mu} = \delta_\mu, \end{cases}$$

welche die Orthogonalität der Vierbeine im Raume mit der Maßbestimmung [42]

$$(96) \qquad g_{\mu\nu} = e_k h_{k,\mu} h_{k,\nu}$$

ausdrücken.

Bedeutet noch

$$(97) \qquad \eta^l_{\varrho\sigma} = \frac{\partial h_{l,\varrho}}{\partial x^\sigma} - \frac{\partial h_{l,\sigma}}{\partial x^\varrho}$$

und

$$(98) \qquad 2\gamma_{mkl} = \underline{\left(\eta^l_{\varrho\sigma} h^\sigma_m h^\varrho_k + \eta^m_{\varrho\sigma} h^\sigma_l h^\varrho_k + \eta^k_{\varrho\sigma} h^\sigma_m h^\varrho_l\right) h'},$$ [43]

ferner

$$(99) \qquad C_l = \frac{1}{4} e_k \alpha_m \alpha_k \gamma_{mkl} + \frac{e}{\omega} \varphi_\sigma h_l^\sigma h', \quad \left(\omega = \frac{h c}{2\pi i}\right)$$ [44]

$$(100) \qquad \gamma^\sigma = e_k \alpha_k h_k^\sigma h',$$ [45]

so lautet der Materieanteil der Lagrangefunktion: [46]

$$(101) \qquad \boldsymbol{R}\, \omega\, \psi^* \left(\gamma^\sigma \frac{\partial \psi}{\partial x^\sigma} - e_l \alpha_l C_l \psi\right) - m c^2 \psi^* \sigma \psi h'.$$

[x^* = komplex konjugiertes von x, $\boldsymbol{R}\,x$ = Realteil von x, $\boldsymbol{I}\,x$ = Imaginärteil von x].

Nun ist [vgl. Fock, a. a. O. Formel (24)]

$$e_l(\alpha_l C_l + C_l^{\dagger}\alpha_l) = -\frac{\partial \gamma^\sigma}{\partial x^\sigma}$$

und infolgedessen

$$\boldsymbol{I}\,\omega\,\psi^*\left(\gamma^\sigma \frac{\partial \psi}{\partial x^\sigma} - e_l\,\alpha_l\,C_l\,\psi\right) = \frac{\omega}{2}\,\frac{\partial}{\partial x^\sigma}(\psi^*\gamma^\sigma\psi). \tag{102}$$

Wir können also für den Materieanteil statt (101) [47]

$$\mathfrak{W} = \omega\,\psi^*\left(\gamma^\sigma \frac{\partial \psi}{\partial x^\sigma} - e_l\,\alpha_l\,C_l\,\psi\right) - m\,c^2\,\psi^*\,\sigma\,\psi\,h' \tag{103}$$

setzen.

Für den Gravitationsanteil nehmen wir $\frac{1}{2\varkappa}\mathfrak{G}$, wo $\varkappa = \frac{8\pi f}{c^4}$ (f = Newtonsche Gravitationskonstante) und [48]

$$\left\{\begin{aligned} \mathfrak{G} = {} & e_k e_l \eta^l_{\varrho\sigma} h_l{}^\varrho h_k{}^{\varrho'} g^{\sigma\sigma'} h' \eta^k_{\varrho'\sigma'} - \tfrac{1}{2} e_k e_l \eta^l_{\varrho\sigma} h_l{}^{\varrho'} h_k{}^\varrho g^{\sigma\sigma'} h' \eta^k_{\varrho'\sigma'} \\ & - \tfrac{1}{4} e_l \eta^l_{\varrho\sigma} g^{\sigma\sigma'} g^{\varrho\varrho'} h' \eta^l_{\varrho'\sigma'}; \end{aligned}\right. \tag{104}$$

wie man leicht nachrechnet (vgl. etwa Weyl, a. a. O.) unterscheidet sich $\mathfrak{G}$ von der skalaren Krümmungsdichte $\mathfrak{R}$ um eine Divergenz:

$$\mathfrak{R} = \mathfrak{G} - 2\,\frac{d}{dx^\nu}\underline{\left(e_l\,h_l{}^\nu\,\frac{\partial(h_l{}^\sigma h')}{\partial x^\sigma}\right)}. \tag{105}$$

Insgesamt ist also

$$\mathfrak{L} = \frac{1}{2\varkappa}\mathfrak{G} + \mathfrak{E} + \mathfrak{W}. \tag{106}$$

Zum Unterschied gegenüber der gewöhnlichen Form der Relativitätstheorie, wo die Feldgrößen $g_{\mu\nu}$ keine Vektoren, sondern Tensoren 2. Stufe waren, sind in (105) die beiden Bestandteile $\mathfrak{G}$ und $2\,\frac{d}{dx^\nu}\underline{\left(e_l\,h_l{}^\nu\,\frac{\partial(h_l{}^\sigma h')}{\partial x^\sigma}\right)}$ von $\mathfrak{R}$ skalare Dichten bezüglich der allgemeinen relativistischen Transformationsgruppe. Dagegen ist nicht $\mathfrak{G}$ allein, sondern erst $\mathfrak{R}$ echt beininvariant. [49]

§ 12. Die Eichinvarianzgruppe

Die einfachste Gruppe, die unsere Funktion $\mathfrak{L}$ gestattet, ist die Eichinvarianzgruppe, bei welcher die x^ν und die Variabeln $h_{i,\nu}$ invariant bleiben, während sich die φ_ν und ψ folgendermaßen transformieren: [50]

$$\text{(107)} \qquad \begin{cases} \delta\varphi_\nu = \dfrac{\partial \xi}{\partial x^\nu}, \\ \delta\psi = -\dfrac{e}{\omega}\,\xi\,\psi. \end{cases}$$

Gegenüber dieser Gruppe ist $\delta\mathfrak{L} = 0$.

Setzen wir, um den Vergleich mit der allgemeinen Theorie zu erleichtern [1]),

$$\varphi_\nu = Q_\nu, \qquad \psi = Q_5,$$

so haben wir

$$c_\nu{}^\mu = \delta_\nu{}^\mu, \qquad c_5{}^\mu = 0$$

und folglich als einzige eigentliche Identität

$$\text{(108)} \qquad \mathfrak{P}^4 = 0.$$

Das folgt natürlich aus der direkten Ausrechnung der $\mathfrak{P}^{\alpha\nu}$:

$$\text{(109)} \qquad \begin{cases} \mathfrak{P}^{\mu\nu} = \mathfrak{F}^{\nu\mu}, \\ \mathfrak{P}^{5\nu} = \omega\,\psi^*\gamma^\nu. \end{cases}$$

Um dieses einfache Beispiel weiter zu diskutieren, sehen wir zunächst von der Gravitation ab, d. h. setzen wir $h_{i,\nu} = \delta_{i\nu}$. [51]

Die Hamiltonfunktion hat dann die Form

$$\text{(110)} \qquad \mathfrak{H} = \mathfrak{H}_0 + \lambda\,\mathfrak{P}^4,$$

wo $\mathfrak{H}_0$ z. B. die in H. P. II gewählte spezielle Hamiltonfunktion ist, welche $\mathfrak{P}^4$ nicht enthält. [52]

Die Feldgleichungen lauten [53]

$$\text{(111)} \qquad \begin{cases} \omega\,\dot{Q}_{\bar\nu} = [\bar{\mathfrak{H}}_0, Q_{\bar\nu}], \\ \quad \dot{Q}_4 = \lambda, \\ \omega\,\dot{Q}_5 = [\bar{\mathfrak{H}}_0, Q_5]; \end{cases}$$

$$\text{(112)} \qquad \omega\,\dot{\mathfrak{P}}^\alpha = [\bar{\mathfrak{H}}_0, \mathfrak{P}^\alpha], \quad (\alpha = 1, \ldots, 5);$$

---

1) Da die ψ nicht hermitisch sind, so sind dem allgemeinen Schema geringe Modifikationen anzubringen, um es auch diesen Variablen anzupassen. Darauf brauchen wir aber nicht näher einzugehen.

da ferner $j = 1$ ist, so haben wir als Nebenbedingung[1]) außer (108) [54]

(113) $$[\overline{\mathfrak{H}}_0, \mathfrak{P}^4] = 0 :$$

Also bleibt in (111) λ wesentlich unbestimmt und die vierte Gleichung (112) wird durch (113) ersetzt.

Die infinitesimale Transformation $\overline{\mathfrak{M}}$ lautet hier:

$$\overline{\mathfrak{M}} = \int \left\{ \frac{\partial \xi}{\partial x^\nu} \mathfrak{E}^{4\nu} - e\, \xi\, \psi^* \gamma^4 \psi \right\} d x^1 d x^2 d x^3$$

oder durch partielle Integration

$$\overline{\mathfrak{M}} = \int \left\{ \frac{\partial \xi}{\partial x^4} \mathfrak{P}^4 - \xi \left[\frac{\partial \mathfrak{E}^{4\nu}}{\partial x^\nu} + e\, \psi^* \gamma^4 \psi \right] \right\} dx^1 dx^2 dx^3 .$$

Die eckige Klammer ist nichts anderes als $1/\omega\, [\overline{\mathfrak{H}}_0, \mathfrak{P}^4]$ oder $\dot{\mathfrak{P}}^4$, so daß

(114) $$\overline{\mathfrak{M}} = \int \left\{ \frac{\partial \xi}{\partial x^4} \mathfrak{P}^4 - \xi \frac{d\, \mathfrak{P}^4}{d x^4} \right\} d x^1 d x^2 d x^3 ,$$

in Übereinstimmung mit (63′).

Nach der allgemeinen Theorie muß $\ddot{\mathfrak{P}}^4 = 0$ vermöge der Feldgleichungen und Identitäten identisch erfüllt sein; das ist in der Tat die Kontinuitätsgleichung der Elektrizität. [55]

§ 13. Die allgemeine relativistische Kovarianz

Bei einer beliebigen Koordinatentransformation [56]

(115) $$\delta x^\nu = \xi^\nu$$

ist

(115′) $$\delta h_{i,\nu} = - h_{i,\mu} \frac{\partial \xi^\mu}{\partial x^\nu} ,$$

ferner

(115″) $$\begin{cases} \delta \varphi_\nu = - \varphi_\mu \dfrac{\partial \xi^\mu}{\partial x^\nu} , \\ \delta \psi = 0 . \end{cases}$$

Die Lagrangefunktion verhält sich dabei wie eine skalare Dichte. [57]

Behalten wir die Bezeichnungen des vorigen Paragraphen für die zu den φ_ν und ψ konjugierten Impulsen bei, und

1) Mit den Bezeichnungen von H. P. II lautet (113) $C = 0$.

stellen wir die zu den $h_{i,\nu}$ gehörigen $\mathfrak{P}^{\alpha\mu}$ durch $\mathfrak{P}_i^{\nu\mu}$ dar, so lauten die uneigentlichen Identitäten (28) im jetzigen Falle:

$$\varphi_\varrho(\mathfrak{E}^{\mu\nu} + \mathfrak{E}^{\nu\mu}) + h_{i,\varrho}(\mathfrak{P}_i^{\nu\mu} + \mathfrak{P}_i^{\mu\nu}) = 0,$$

mit Rücksicht darauf, daß $\mathfrak{E}^{\mu\nu} + \mathfrak{E}^{\nu\mu} = 0$ ist, reduzieren sie sich zu [58]

$$\mathfrak{P}_i^{\nu\mu} + \mathfrak{P}_i^{\mu\nu} = 0 \tag{116}$$

und haben also die vier eigentlichen Identitäten

$$\mathfrak{P}_i^{4} = 0 \tag{117}$$

zur Folge.

Die direkte Berechnung ergibt in der Tat (116), da $\mathfrak{G}$ und $\mathfrak{W}$ von den $h_{i,\nu,\mu}$ nur durch die $\eta^i_{\tau\sigma}$ abhängen und

$$\frac{\partial \eta^i_{\varrho\sigma}}{\partial h_{i,\nu,\mu}} = \delta^\nu_\varrho \delta^\mu_\sigma - \delta^\mu_\varrho \delta^\nu_\sigma,$$

d. h. antisymmetrisch in μ und ν ist. Man findet [59][60]

$$\left\{\begin{aligned} \mathfrak{P}_i^{\nu\mu} = \Big[\eta^i_{\varrho\sigma} g^{\nu\sigma} g^{\mu\varrho} &+ 2e_l \eta^l_{\varrho\sigma} h^\varrho_l \left(g^{\sigma\mu} h^\nu_i - g^{\sigma\nu} h^\mu_i\right) \\ &- e_l \eta^l_{\varrho\sigma} h^\varrho_i \left(h^\nu_l g^{\sigma\mu} - h^\mu_l g^{\sigma\nu}\right)\Big] e_i h' \cdot \frac{1}{2\varkappa} \\ &- \boldsymbol{R}\frac{\omega}{4}\psi^* \alpha_l \alpha_m \alpha_k \psi e_l e_k \frac{\partial \gamma_{mkl}}{\partial h_{i,\nu,\mu}}. \end{aligned}\right. \tag{118}$$

Die infinitesimale Transformation $\overline{\mathfrak{M}}$ nimmt die Form an: [61]

$$\overline{\mathfrak{M}} = -\int dx^1 dx^2 dx^3 \left\{\frac{\partial \xi^\mu}{\partial x^4}(h_{i,\mu}\mathfrak{P}_i^4 + \varphi_\mu \mathfrak{P}^4) - \xi^\mu \left[\frac{\partial}{\partial x^{\bar\nu}}(h_{i,\mu}\mathfrak{P}_i + \varphi_\mu \mathfrak{P}^{\bar\nu}) + \mathfrak{G}_\mu\right]\right\};$$

betrachten wir insbesondere die Translation $\xi^\mu = \varepsilon^\mu = \text{const}$ so liefert $\frac{d\overline{\mathfrak{M}}}{dx^4} = 0$ den Energieimpulssatz

$$\overline{\mathfrak{G}}_\mu = \text{const};$$

für eine lineare Transformation, bekommen wir daraus eine Verallgemeinerung der Drehimpulssätze (vgl. H. P. II, S. 177).

Nach der allgemeinen Theorie (§ 7) liefert das Nullsetzen des Koeffizienten von ξ^μ eine Nebenbedingung:

$$\mathfrak{G}_\mu + \frac{\partial}{\partial x^{\bar\nu}}(h_{i,\mu}\mathfrak{P}_i^{\bar\nu} + \varphi_\mu \mathfrak{P}^{\bar\nu}) = 0. \tag{119}$$

F. Klein[1]) hat bereits in anderem Zusammenhang bemerkt, daß (119) mit 4 Feldgleichungen äquivalent ist.

§ 14. Die echte Beinkovarianz

Für diese Gruppe ist [62]

$$(120)\quad \begin{cases} \delta x^\nu = 0, & \delta \varphi_\mu = 0, \\ \delta h_{i,\nu} = e_k \xi_{ik} h_{k,\nu}. & (\xi_{ik} = -\xi_{ki}) \end{cases}$$

und, wie man auf Grund von (120) leicht findet, [63]

$$(120')\quad \delta\psi = \tfrac{1}{4} e_k \xi_{ik} \alpha_i \alpha_k \psi .$$

Hier haben wir ein Beispiel des im § 9 behandelten „zweiten Falles" vor uns. Denn $\mathfrak{G}$ ist nur lokalbeininvariant und erst $\mathfrak{R}$ ist echt beininvariant; $\mathfrak{W}$ und $\mathfrak{E}$ sind echt beininvariant.

Um nach Formel (74) $\mathfrak{J}_r^{\,\nu} \equiv \mathfrak{J}_{(ik)}^{\nu}$ auszurechnen, ist es am zweckmäßigsten, vorübergehend als Variable

$$Q_l^{\,\alpha} \equiv h' h_l^{\,\alpha}$$

zu wählen. Dann ist nach (105) [64]

$$\mathfrak{f}_l^{\nu,\alpha\varrho} = -\frac{1}{\varkappa} e_l h_l^{\,\nu} \delta_\varrho^{\,\alpha}$$

und nach (120) [65]

$$c_{l,\alpha(ik)} = \delta_{il} e_k h_k^{\,\alpha} h' - \delta_{kl} e_i h_i^{\,\alpha} h'.$$

Unter Berücksichtigung von (71) findet man daraus leicht [66]

$$(121)\quad \mathfrak{J}_{(ik)}^{\nu} = \frac{1}{\varkappa} e_i e_k \frac{d}{dx^\varrho} (h' h_i^{\,\nu} h_k^{\,\varrho} - h' h_i^{\,\varrho} h_k^{\,\nu}).$$

Zur Berechnung von $\mathfrak{F}_r \equiv \mathfrak{F}_{(ik)}$ ist es bequemer, zu den ursprünglichen Variablen $Q_{l,\alpha} \equiv h_{l,\alpha}$ und $Q_5 = \psi$ zurückzukehren. Dann ist [67]

$$c_{l,\alpha(ik)} = \delta_{il} e_k h_{k,\alpha} - \delta_{kl} e_i h_{i,\alpha},$$

ferner nach (120') [68]

$$c_{5(ik)} = \tfrac{1}{4}(e_k \alpha_i \alpha_k \psi - e_i \alpha_k \alpha_i \psi)$$

zu setzen. Danach ist [69]

$$\mathfrak{F}_{(ik)} = \underline{\mathfrak{P}_i^{\,\nu} e_k h_{k,\nu} - \mathfrak{P}_k^{\,\nu} e_i h_{i,\nu}} + \frac{\omega}{4} e_k e_l h' h_l^{\,4} \psi^* (\alpha_l \alpha_i \alpha_k - \alpha_l \alpha_k \alpha_i) \psi.$$

---

1) Gött. Nachr. 1918, S. 185.

Nun ist nach (98)

$$\frac{\partial \gamma_{mjl}}{\partial h_{i,\nu,4}} e_k h_{k,\nu} - \frac{\partial \gamma_{mjl}}{\partial h_{k,\nu,4}} e_i h_{i,\nu} = h' h_l^4 (\delta_{im}\delta_{kj} - \delta_{ij}\delta_{km});$$

setzt man also gemäß (118)

$$\mathfrak{P}_i^\nu = \tilde{\mathfrak{P}}_i^\nu - \boldsymbol{R}\frac{\omega}{4} e_l \psi^* \alpha_l \alpha_m \alpha_j \psi e_j \frac{\partial \gamma_{mjl}}{\partial h_{i,\nu,4}}, \tag{122}$$

wo also $\tilde{\mathfrak{P}}_i^\nu$ die Impulse bei Abwesenheit von Materie darstellen, so reduzieren sich die $\mathfrak{J}_{(ik)}$ auf

$$\mathfrak{J}_{(ik)} = \underline{\tilde{\mathfrak{P}}_i^\nu e_k h_{k,\nu} - \tilde{\mathfrak{P}}_k^\nu e_i h_{i,\nu}}, \tag{123}$$

wie es auch sein muß.

Gemäß (121) und (123) lauten die sechs eigentlichen Identitäten

$$\underline{\tilde{\mathfrak{P}}_i^\nu e_k h_{k,\nu} - \tilde{\mathfrak{P}}_k^\nu e_i h_{i,\nu}} + \frac{1}{\varkappa} e_i e_k \frac{d}{dx^\varrho}(h' h_i^4 h_k^\varrho - h' h_i^\varrho h_k^4) = 0, \tag{124}$$

die sich auch direkt aus (118) ergeben.

§ 15. Ergänzende Bemerkungen über das Gravitations- und Materiefeld

1. Nachdem wir in den vorigen Paragraphen skizziert haben, wie die Fock-Weylsche Theorie des Einkörperproblems gequantelt werden kann, möchten wir auf einen Punkt dieses Einkörpermodells kurz eingehen, der bei Fock und bei Weyl verschieden behandelt ist, nämlich die Aufstellung des Impulsenergietensors $\mathfrak{T}_i^\nu$ der Materie. Der Focksche Ansatz, der zu einem unsymmetrischen Tensor führt, scheint uns unzweckmäßig und wir ziehen die Weylsche Definition

$$\mathfrak{T}_i^\nu = \frac{\delta \mathfrak{W}}{\delta h_{i,\nu}} \tag{125c}$$ [70]

vor, da sie auf Grund der Feldgleichungen einen symmetrischen Tensor liefert. Da aber Weyl mit einem zweikomponentigen ψ operiert, während wir mit Fock bei der Vierkomponententheorie bleiben wollen, so wird es wohl nicht überflüssig sein, die Weylsche Berechnung von $\mathfrak{T}_i^\nu$ hier mutatis mutandis zu wiederholen.

Die Symmetrie von $\mathfrak{T}_i^{\,\nu}$ folgt unmittelbar aus $\delta\,\mathfrak{W} = 0$, wo δ die Variation (120), (120′) ist. Denn man bekommt daraus durch Nullsetzen des Koeffizienten von $\mathfrak{x}_{ik}$

$$\mathfrak{T}_i^{\,\nu} e_k h_{k,\nu} - \mathfrak{T}_k^{\,\nu} e_i h_{i,\nu} = -\frac{1}{2}\,\boldsymbol{R}\,\frac{\delta\,\mathfrak{W}}{\delta\,\psi}\,(e_k\,\alpha_i\,\alpha_k - e_i\,\alpha_k\,\alpha_i)\,\psi, \qquad [71]$$

d. h.

$$\mathfrak{T}_i^{\,\nu}\, e_k\, h_{k,\nu} - \mathfrak{T}_k^{\,\nu}\, e_i\, h_{i,\nu} = 0 \qquad [72]$$

auf Grund der Feldgleichungen

$$\frac{\delta\,\mathfrak{W}}{\delta\,\psi} = 0 \quad \text{und} \quad \frac{\delta\,\mathfrak{W}}{\delta\,\psi^*} = 0.$$

Diese Gleichung drückt aus, daß der Tensor

$$\mathfrak{T}''_{ik} = e_i\, e_k\, \mathfrak{T}_i^{\,\nu}\, h_{k,\nu} \qquad [73]$$

symmetrisch in bezug auf i und k ist.

Statt (125c) können wir ebensogut

$$\text{(126c)} \qquad \mathfrak{T}_i^{\,\nu} = \frac{\delta\,\boldsymbol{R}\,\mathfrak{W}}{\delta\,h_{i,\nu}}$$

setzen, was uns einen reellen Tensor $\mathfrak{T}_i^{\,\nu}$ ergeben wird. Bequemer berechnen wir

$$\text{(127c)} \qquad \mathfrak{T}'_{i,\nu} = \frac{\delta\,\boldsymbol{R}\,\mathfrak{W}}{\delta\,h_i^{\,\nu}} = -\,\mathfrak{T}_k^{\,\varrho}\, e_k\, h_{k,\nu}\, h_{i,\varrho} \equiv e_i\, h'\, T'_{i,\nu}. \qquad [74]$$

Auf Grund von (103) finden wir

$$\text{(128c)} \left\{ \begin{aligned} T'_{i,\nu} = {} & \boldsymbol{R}\,\omega\,\psi^*\,\alpha_i\,\frac{\partial\,\psi}{\partial x^\nu} - e\,\psi^*\,\alpha_i\,\psi\,\varphi_\nu - h_{i,\nu} W \qquad [75] \\ & + \boldsymbol{R}\,\frac{\omega}{4}\,e_k\,h_k^{\,\varrho}\,h_{m,\nu}\,\frac{\partial}{\partial x^\varrho}\,\{\psi^*\,\alpha_i\,\alpha_m\,\alpha_k\,\psi\} \\ & - \boldsymbol{R}\,\frac{\omega}{4}\,e_i e_l e_k\,\psi^*\,\alpha_l\,\alpha_m\,\alpha_k\,\psi \left\{\frac{\partial\,\gamma_{mkl}}{\partial h_i^{\,\nu}} - \frac{\partial}{\partial x^\varrho}\,\frac{\partial\,\gamma_{mkl}}{\partial h^{\nu}_{i,\varrho}}\right\}, \\ & \text{mit } W = \frac{1}{h'}\,\mathfrak{W}. \end{aligned} \right.$$

Beschränken wir uns auf die spezielle Relativität, indem wir

$$h_i^{\,\nu} = e_i\, h_{i,\nu} = \delta_{i\nu} \qquad [76]$$

setzen. Dann wird (128c)[1])

1) Vgl. auch H. **Tetrode**, Ztschr. f. Phys. **49**. S. 858. 1928 Formeln (13) und (16), sowie den Text auf S. 862.

$$\text{(129c)} \begin{cases} T'_{i,\nu} = \boldsymbol{R}\,\omega\,\psi^*\,\alpha_i\,\dfrac{\partial\psi}{\partial x^\nu} - \delta_{i\nu}W - e\,\psi^*\,\alpha_i\,\psi\,\varphi_\nu \\ \qquad + \boldsymbol{R}\,\dfrac{\omega}{4}\,e_\varrho\,e_\nu\,\dfrac{\partial}{\partial x^\varrho}\,(\psi^*\,\alpha_i\,\alpha_\nu\,\alpha_\varrho\,\psi), \\ \text{mit} \\ W = \boldsymbol{R}\,\omega\,e_\varrho\,\psi^*\,\alpha_\varrho\,\dfrac{\partial\psi}{\partial x^\varrho} - m\,c^2\,\psi^*\sigma\,\psi - e_o e\,\psi^*\alpha_o\psi\,\varphi_\varrho. \end{cases}$$ [77]

Insbesondere ist dann

$$T'_{44} = \boldsymbol{R}\,\omega\,\psi^*\,\alpha_{\bar\varrho}\,\frac{\partial\psi}{\partial x^{\bar\varrho}} - e\,\psi^*\,\alpha_{\bar\varrho}\,\psi\,\varphi_{\bar\varrho} + m\,c^2\,\psi^*\,\sigma\,\psi,$$ [78]

d. h. für den Energieoperator

$$\text{(130c)} \qquad H = \alpha_{\bar\varrho}\left(\frac{h}{2\pi i}\,\frac{\partial}{\partial x^{\bar\varrho}} - \frac{e}{c}\,\varphi_{\bar\varrho}\right) + m\,c\,\sigma.$$ [79]

Ferner ist [80]

$$T'_{4\bar\nu} = \boldsymbol{R}\,\omega\,\psi^*\,\frac{\partial\psi}{\partial x^{\bar\nu}} - e\,\psi^*\,\psi\,\varphi_{\bar\nu} + \boldsymbol{R}\,\frac{\omega}{4}\,\frac{\partial}{\partial x^{\bar\varrho}}\,(\psi^*\,\alpha_{\bar\nu}\,\alpha_{\bar\varrho}\,\psi);$$

setzen wir

$$\text{(131)} \qquad \alpha_1\,\alpha_2 = \mu_3$$

und zyklisch, so wird z. B.

$$T'_{41} = \boldsymbol{R}\,\omega\,\psi^*\,\frac{\partial\psi}{\partial x^1} - e\,\psi^*\,\psi\,\varphi_1 + \frac{\omega}{4}\left\{\frac{\partial}{\partial x^2}\,(\psi^*\,\mu_3\,\psi) - \frac{\partial}{\partial x^3}\,(\psi^*\,\mu_2\,\psi)\right\}.$$

Der Impulsoperator lautet danach:

$$\text{(132c)} \qquad p_{\bar\nu} = \frac{h}{2\pi i}\,\frac{\partial}{\partial x^{\bar\nu}} - \frac{e}{c}\,\varphi_{\bar\nu};$$

andererseits bekommt man für den Drehimpuls:

$$\begin{aligned} M_1 &= x^2\,T'_{43} - x^3\,T'_{42} \\ &= \boldsymbol{R}\,\omega\,\psi^*\left(x^2\,\frac{\partial}{\partial x^3} - x^3\,\frac{\partial}{\partial x^2}\right)\psi - e\,\psi^*\,\psi\,[x^2\,\varphi_3 - x^3\,\varphi_2) \\ &+ \frac{\omega}{4}\left\{x^2\,\frac{\partial}{\partial x^1}\,(\psi^*\,\mu_2\,\psi) - x^2\,\frac{\partial}{\partial x^2}\,(\psi^*\,\mu_1\,\psi) - x^3\,\frac{\partial}{\partial x^3}\,(\psi^*\,\mu_1\,\psi) \right. \\ &\left. + x^3\,\frac{\partial}{\partial x^1}\,(\psi^*\,\mu_3\,\psi)\right\}, \end{aligned}$$

folglich für den entsprechenden Operator

$$(133c)\quad \begin{cases} M_1 = \dfrac{h}{2\pi i}\left(x^2 \dfrac{\partial}{\partial x^3} - x^3 \dfrac{\partial}{\partial x^2}\right) \\ \qquad\qquad - \dfrac{e}{c}(x^2 \varphi_3 - x^3 \varphi_2) + \dfrac{i\mu_1}{2}\,. \end{cases}$$

2. Im Vorangehenden haben wir für die Vierbeine $h_{i,\nu}$ die Bose-Einsteinsche Statistik angenommen, d. h. die Klammersymbole in den V.-R. mit dem Minuszeichen gewählt. Nun fragt es sich, ob es möglich wäre, auch die Fermische Statistik auf die Vierbeine anzuwenden. Das Kriterium für die Zulässigkeit der V.-R. mit dem Pluszeichen ist folgendes (vgl. H.P. I, S. 29): Es sollen die *gewöhnlichen* Klammersymbole (mit dem Minuszeichen) $[\mathfrak{G}_\mu, Q_\alpha]$, $[\mathfrak{G}_\mu, \mathfrak{P}^\alpha]$ denselben Wert behalten, wenn man in $[Q_\alpha, Q_\beta]$, $[\mathfrak{P}^\alpha, \mathfrak{P}^\beta]$ und $[Q^\alpha, \mathfrak{P}^\beta]$ das Minuszeichen durch das Pluszeichen ersetzt.

Nach diesem Kriterium ist aber die Frage bezüglich der Vierbeine zu *verneinen*. Denn man sieht aus der Form der Hamiltonfunktion (quadratisch in den $\mathfrak{P}^\alpha$), daß $[\mathfrak{H}_0, Q_\alpha]$ beim Übergang vom Plus- zum Minuszeichen eine Änderung erfährt: die zum in $\mathfrak{P}^\alpha$ quadratischen Anteil von $\mathfrak{H}_0$ gehörigen Klammersymbole sind nämlich in beiden Fällen verschieden und die Differenzen kompensieren sich nicht.

3. Das reine (Vakuum-)Gravitationsfeld ließe sich durch die $g_{\mu\nu}$ statt durch die $h_{i,\nu}$ beschreiben. Dann hätten wir mit einer etwas anderen Abart des „zweiten Falles" zu tun und bekämen wegen der allgemeinen Kovarianzgruppe vier Identitäten von der Gestalt $(\mathfrak{P}^\alpha + \mathfrak{R}^{\alpha 4})\, c^4_{\alpha r} = 0$.

Zusammenfassung

1. Verhält sich die Lagrangefunktion $\mathfrak{L}(Q_\alpha; \dot{Q}_\alpha)$ gegenüber der Gruppe[1])

$$(2')\quad \begin{cases} \delta x^\nu = a_r^{\nu,0}(x)\, \xi^r(x), \\ \delta Q_\alpha = c^0_{\alpha r}(x, Q)\, \xi^r + c^\sigma_{\alpha r} \dfrac{\partial \xi^r}{\partial x^\sigma} \end{cases}$$

1) Der Übersichtlichkeit halber spezialisieren wir hier die Formeln auf den physikalisch wichtigen Fall $j = 1$.

wie eine skalare Dichte, so bestehen zwischen den Q und konjugierten Impulsen $\mathfrak{P}$ die Identitäten

$$(29') \qquad \mathfrak{F}_r \equiv \underline{\mathfrak{P}^\alpha c^4_{\alpha\, r}} = 0\,.$$

Falls nicht $\mathfrak{L}$, sondern $\mathfrak{L} + \mathfrak{L}'$ eine skalare Dichte ist, wobei $\mathfrak{L}'$ die zweiten Ableitungen der Q_α linear enthält, so tritt überall $\mathfrak{P}^\alpha + \mathfrak{R}^{\alpha 4}$ an Stelle von $\mathfrak{P}^\alpha$.

2. Infolgedessen ergibt die Auflösung der Gleichungen

$$\mathfrak{P}^\alpha = \frac{\partial \mathfrak{L}}{\partial \dot{Q}_\alpha}$$

nach den $\dot{Q}_\alpha$:

$$(31') \qquad \dot{Q}_\alpha = \dot{Q}^0_\alpha(\mathfrak{P}, Q) + \lambda^r c^4_{\alpha\, r}\,,$$

mit willkürlichen Raumzeitfunktionen λ^r.

Die Hamiltonfunktion nimmt sodann die Form

$$(35') \qquad \mathfrak{H} = \mathfrak{H}_0(\mathfrak{P}, Q) + \lambda^r \mathfrak{F}_r$$

an. Die Grundgleichungen der Theorie sind die kanonischen Feldgleichungen, die kanonischen V.-R., die Nebenbedingungen

$$\mathfrak{F}_r = 0 \quad \text{und} \quad \frac{d\mathfrak{F}_r}{dx^4} = 0\,.$$

3. Die infinitesimale Transformation der Gruppe läßt sich darstellen durch

$$(45) \qquad \omega\, \delta^* \Phi = [\overline{\mathfrak{M}}, \Phi],$$

$$(46) \qquad \mathfrak{M} = \underline{\mathfrak{P}^\alpha\, \delta Q_\alpha} - \mathfrak{G}_\mu\, \delta x^\mu\,.$$

[Φ beliebiges, nur von Q und $\mathfrak{P}$ abhängiges Funktional; $\mathfrak{G}_\mu$ Impulsenergie(pseudo)dichte].

Ein Spezialfall von $\overline{\mathfrak{M}}$ auf einem beliebigen Schnitt $x^4 = x_0{}^4$ ist $\overline{\varepsilon^r \mathfrak{F}_r}$. Daraus folgt, daß die $\mathfrak{F}_r$ auf Grund von $\mathfrak{F}_r = 0$ untereinander vertauschbar, d. h. die Nebenbedingungen $\mathfrak{F}_r = 0$ verträglich sind.

Ferner ist vermöge der Feldgleichungen

$$(58) \qquad \frac{d\overline{\mathfrak{M}}}{dx^4} = 0\,,$$

woraus folgt

$$(63') \qquad \overline{\mathfrak{M}} = \int dx^1\, dx^2\, dx^3 \left\{ \mathfrak{F}_r \frac{\partial \xi^r}{\partial x^4} - \frac{d\mathfrak{F}_r}{dx^4}\, \xi^r \right\}$$

und

$$\frac{d^2 \mathfrak{F}_r}{(d x^4)^2} = 0 \tag{64'}$$

auf Grund der Feldgleichungen (zeitliche Fortpflanzung der Nebenbedingungen).

4. Das Schema der Grundgleichungen bleibt gegenüber der Gruppe invariant.

5. Als Beispiele werden das elektromagnetische Feld, das Diracsche Materiefeld und das Gravitationsfeld samt deren Wechselwirkungen behandelt. Die dabei in Betracht kommenden Gruppen sind die Eichinvarianzgruppe, die echte Beinkovarianzgruppe und die Gruppe der allgemeinen Relativitätstheorie.

Was insbesondere die Gravitation betrifft, so ist es unmöglich, die betreffenden Feldgrößen gemäß der Fermischen Statistik zu quanteln.

Hrn. Prof. Pauli spreche ich meinen aufrichtigen Dank aus für die Anregung zu dieser Arbeit und seine wertvollen Ratschläge.

Zürich, Physik. Institut der Eidgen. Technischen Hochschule, den 5. März 1930.

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Translation and Commentary of Leon Rosenfeld's "Zur Quantelung der Wellenfelder", *Annalen der Physik* 397, 113 (1930), by Donald Salisbury, Max-Planck-Institut für Wissenschaftsgeschichte, Berlin and Austin College, Sherman, TX, USA

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On the quantization of wave fields

L. Rosenfeld

Introduction

Heisenberg and Pauli [1] have recently made substantial progress in the formulation of the general quantum mechanical laws for electromagnetic and material wave fields through the systematic development of Dirac's method of second quantization. In addition to certain deeper lying technical difficulties a characteristic difficulty of a more formal nature appeared; the momentum conjugate to the scalar potential vanishes identically. The formation of the Hamilton function and the commutation relations cannot be carried out without further work. Three methods have been proposed to date to resolve these problems. They do, to be sure, fulfill their objective but they can hardly be viewed as satisfactory.

1. The first Heisenberg-Pauli method is a purely analytical artifice.[2] New terms are added to the Lagrange function, multiplied by a small parameter ϵ. These have the effect that the above-mentioned momentum no longer vanish. In the final result one then takes the limit $\epsilon = 0$. However, the ϵ-terms lead to unphysical caculational complications and destroy the characteristic invariance of the Lagrangian under the gauge invariance group.

2. The second Heisenberg-Pauli method [3] uses this invariance in an essential way. The scalar potential is given a certain arbitrary value, e.g. zero; then the Hamiltonian method delivers one less equation of motion. Supposing the missing equation of motion is $C = 0$, then one finds as a consequence of the gauge invariance of the Hamiltonian that $C =$ constant. The choice of the value $C = 0$ for this constant signifies a restriction to a system of terms that do not depend on each other. But distinguishing a component of the 4-potential

[1] W. Heisenberg and W. Pauli, Zeit. f. Phys. **54**, 1 (1929); **59**, 168 (1930). In the following referred to as H. P. I and H. P. II.

[2] H.P.I, pages 24-29, 30 ff.

[3] cf. L. Rosenfeld, Zeit. f. Phys. **58**, 540 (1929)

necessitates a proof of the relativistic covariance of this method, and this check is very troublesome.

3. The Fermi method [4] consists also in adding terms to the Lagrangian in such a manner that no momentae vanish identically. In order that the resulting field equations agree with the usual equations certain constraints must be fulfilled. Then it must be shown that when these constraints hold on a $t =$ constant slice they continue to hold under propagation in time. The disadvantage of this method is that once again the gauge invariance is destroyed.

The identical vanishing of the cited momentum component is by no means an isolated phenomenon; the origin is the gauge invariance of the Lagrangian as is shown below in a simple, comprehensive discussion. In an analogous fashion, i.e., generally, the appearance of identical relations between variables and conjugate momentae is to be expected in all cases in which the Lagrangian permits a suitably built group. As I was investigating these relations in the especially instructive example of gravitation theory, Professor Pauli helpfully indicated to me the principles of a simpler and more natural manner of applying the Hamiltonian procedure in the presence of identities. This procedure is not subject to the disadvantages of the earlier methods. In the following the subject will be treated first from a general group theoretical perspective, and then illustrated with various physical examples. [5]

1. Part One: General Theory

§1. Assumptions about the Lagrange function and the underlying group

We consider any dynamical system defined through the field quantities $Q_\alpha(x^1, x^2, x^3, x^4)$ that depend on the spatial coordinates x^1, x^2, x^3 and the time coordinate $x^4 = ct$ (and not, as in H. P., $x^4 = ict$!). We need to make no assumptions about the Lagrange function $\mathcal{L}\left(Q; \frac{\partial Q}{\partial x}\right)$ as long as we remain in the framework of the classical theory, i.e., we work only with c-numbers. If we were to consider the Q variables as q-numbers (while the spacetime coordinates remain c-numbers) then we would have to take into account that the rule for the derivative of the function of a function would lose its general validity. [6] If we want to keep certain properties of the Lagrange function that follow from this rule (and this will be the case) then it will be necessary to make certain restrictive assumptions that preserve these properties in spite of the failure of the derivative rule. It turns out that from the mathematical point of view these restrictions will be extensive, though for physically interesting Lagrangians they are fulfilled. They concern the factor ordering of non-commuting terms, and the analytical behav-

[4] cf. H. P. II, page 175, footnote

[5] Here I want to stress once and for all that in the special cases treated in the works H. P. I and H. P. II the path to the desired generalization frequently suggested itself to me. It would serve little purpose in the following to refer to each instance.

[6] cf. H. P. I, p. 18, further p. 14, footnote 1.

ior of the Lagrange function; it must be at most quadratic in the derivatives of the Q.

To abbreviate will often write $Q_{\alpha,\nu}$ instead of $\frac{\partial Q_\alpha}{\partial x^\nu}$, and also $\dot{Q}_\alpha$ instead of $Q_{m,4} \equiv \frac{\partial Q_\alpha}{\partial x^4}$. Furthermore we will suppress the summation sign following the familiar rule. With these choices the Lagrange function takes the form

$$2\mathcal{L} = Q_{\alpha,\nu}\mathcal{A}^{\alpha\nu,\beta\mu}(Q)Q_{\beta,\mu} + Q_{\alpha,\nu}\mathcal{B}^{\alpha\nu}(Q) + \mathcal{B}^{\alpha\nu}(Q)Q_{\alpha,\nu} + \mathcal{C}(Q). \tag{1}$$

Although only the $\dot{Q}_\alpha$ do not commute with the Q_α, we must nevertheless retain a fixed factor ordering for the remaining derivatives since certain operations, $\frac{d}{dx^4}$ for example, convert variables to others that no longer commute. Therefore the result of such operations depends on the original factor ordering.

Since c-number considerations are often superior from the point of view of generality and elegance we will for the moment in this first overview use them. Later we will indicate the appropriate modifications required for q-numbers. However, to avoid unnecessary repetition we will refer to commutation relations for c-numbers, whereby we naturally mean the corresponding Poisson brackets.

We turn now to the definition of the transformation group that the Lagrangian function permits (the precise sense to be specified shortly). We are in no way attempting in this investigation to address the most general situation. Rather, we seek a description that is sufficiently general that in the physical applications the deeper interconnections are clearly evident. We do not therefore ask for the most general Lagrangian from which identities of the type mentioned above will result. Rather we will take as our basis a special yet extended class of continuous infinite groups. We show that they lead to identities for arbitrary c-number Lagrangian functions. [7]

We characterize our group through its infinitesimal transformations. We assume that both the x^ν and the Q_α transform in a certain way. Indeed, the δx^ν (respectively the δQ_α) depend on r_0 arbitrary real functions $\xi^r(x)\,(r = 1, 2, \cdots r_0)$ and their derivatives up to order k (respectively j); the coefficients of these derivatives must be real, and (here is the group specialization) the δx^ν depend only on x^ν while the δQ_α depend only on x^ν and Q_α, and not on derivatives of the Q_α. Explicitly,[1]

$$\left\{ \begin{aligned} \delta x^\nu &= a_r^{\nu,0}(x)\xi^r(x) + a_r^{\nu,\sigma}(x)\frac{\partial \xi^r}{\partial x^\sigma} + a_r^{\nu,\sigma\cdots\tau}(x)\frac{\partial^k \xi^r}{\partial x^\sigma \cdots \partial x^\tau}, \\ \delta Q_\alpha &= c^0_{\alpha r}(x,Q)\xi^r(x) + c^\sigma_{\alpha r}(x,Q)\frac{\partial \xi^r}{\partial x^\sigma} + c^{\sigma\cdots\tau}_{\alpha r}(x,Q)\frac{\partial^j \xi^r}{\partial x^\sigma \cdots \partial x^\tau}. \end{aligned} \right. \quad [2] \tag{2}$$

In addition we have the essential assumption, [8]

$$j \geq k + 1. \tag{3}$$

Concerning the commutation relations relating to the functions that appear in (2), the ξ^r must be c-numbers, and this property is preserved (corresponding to

[7] The method that is used here furthermore gives an immediate response to the general question that was just posed. In specially constructed Lagrange functions the group does not even need to be infinite in order that identities result.

[8] We set $\frac{\partial^0 \xi}{(\partial x)^0} \equiv \xi$ and $\frac{\partial^{-1} \xi}{(\partial x)^{-1}} \equiv 0$. [3]

the fact that the ξ^r depend only on the x^ν). Since the a depend only on the x^ν we may also consider them to be c-numbers. Then the δx^ν are also c-numbers, as they must be in order that we may treat the x^ν as c-numbers.

The most important groups appearing in physics are of this type (cf. the second part of this work).

It remains for us to express the fact that the integral

$$\int \mathcal{L} dx^1 dx^2 dx^3 dx^4,$$

is invariant under the transformations (2). For this purpose we introduce a few concepts.

Besides the "local" variations $\delta\Phi(x, Q, \frac{\partial Q}{\partial x}, \ldots)$ we have the "substantial" variation [4]

$$\delta^* \Phi = \delta\Phi - \frac{d\Phi}{dx^\nu}\delta x^\nu; \tag{4}$$

if we represent transformed quantities with a prime, then we have

$$\delta\Phi = \Phi'[x'; Q'(x'); \cdots] - \Phi[x; Q(x); \cdots],$$

while

$$\delta^*\Phi = \Phi'[x; Q'(x); \cdots] - \Phi[x; Q(x); \cdots],$$

The following important formulae result (also for q-numbers):

$$\delta^* \frac{d\Phi}{dx^\nu} = \frac{d}{dx^\nu}\delta^*\Phi, \tag{5}$$

and [5]

$$\delta \frac{d\Phi}{dx^\nu} = \frac{d}{dx^\nu}\delta\Phi - \frac{d\Phi}{dx^\rho}\frac{d\delta x^\rho}{dx^\nu}. \tag{6}$$

A quantity $\mathbf{R}$ is called a scalar density (with respect to the group) when it transforms with the following properties:

$$\delta^*\mathbf{R} + \frac{d}{dx^\nu}\left(\mathbf{R}\delta x^\nu\right) = 0, \tag{7}$$

or, according to (4), [6]

$$\delta\mathbf{R} + \mathbf{R}\frac{d\delta x^\nu}{dx^\nu} = 0. \tag{8}$$

Quantities depend in general on two kinds of indices, first on indices $\alpha, \beta, \gamma, \cdots$ whose range is that of the index α in Q_α. Secondly, they depend on the indices $\mu, \nu, \cdots$ which as with the index of x^ν range from 1 to 4. In particular the index r in ξ^r represents one or more systems of indices of the form

$\{\alpha, \beta, \cdots; \mu, \nu, \cdots\}$numbered in an arbitrary one-dimensional order. The indices of type $\alpha, \beta, \cdots$ could in their turn be multiple and in particular contain systems of indices $\mu, \nu, \cdots$.

A contravariant tensor $K^{\alpha\nu}$ is defined through the transformation property

$$\delta K^{\alpha\nu} = K^{\alpha\mu}\frac{d\delta x^\nu}{dx^\mu} - \underline{K^{\beta\nu}\frac{\partial \delta Q_\beta}{\partial Q_\alpha}}; \tag{9}$$

in the q-number case this definition contains an arbitrariness in the underlined term that we will fix by setting

$$\underline{K^{\beta\nu}\frac{\partial \delta Q_\beta}{\partial Q_\alpha}} = \frac{1}{2}\left(K^{\beta\nu}\frac{\partial \delta Q_\beta}{\partial Q_\alpha} + \frac{\partial \delta Q_\beta}{\partial Q_\alpha}K^{\beta\nu}\right),$$

where $x^\dagger$ is the Hermitian conjugate (adjoint) to x. [In the following we will use the notation

$$\underline{x} = \frac{1}{2}(x + x^\dagger).]$$

With this assignment a Hermitian tensor remains a Hermitian tensor under the transformation group.

A covariant tensor $K_{\alpha\nu}$ has the transformation property

$$\delta K_{\alpha\nu} = -K_{\alpha\mu}\frac{d\delta x^\mu}{dx^\nu} + \underline{K_{\beta\nu}\frac{\partial \delta Q_\alpha}{\partial Q_\beta}}; \tag{10}$$

the variation of mixed tensors $K_{\alpha\beta\cdots}{}^{\gamma\delta\cdots}{}_{\mu\nu\cdots}{}^{\rho\pi\rho}$ is formed in analogy with (9) and (10).

A tensor density $\mathbf{R}^{\alpha\nu}$ transforms as the product of a tensor $K^{\alpha\nu}$ with a scalar density $\mathbf{R}$, namely

$$\delta \mathbf{R}^{\alpha\nu} = \mathbf{R}^{\alpha\mu}\frac{d\delta x^\nu}{dx^\mu} - \underline{\mathbf{R}^{\beta\nu}\frac{\partial \delta Q_\beta}{\partial Q_\alpha}} - \mathbf{R}^{\alpha\nu}\frac{d\delta x^\mu}{dx^\mu}. \tag{11}$$

We are now in position to formulate the invariance condition with respect to the Lagrangian function. In order that the integral $\int \mathcal{L} dx^1 dx^2 dx^3 dx^4$ be invariant we conclude, as is well known [9], that $\mathcal{L}$ must be a scalar density up to a divergence $\mathcal{L}' \equiv \frac{d\mathbf{R}^\nu}{dx^\nu}$. Explicitly:

$$\delta(\mathcal{L} + \mathcal{L}') + (\mathcal{L} + \mathcal{L}')\frac{d\delta x^\nu}{dx^\nu} = 0. \tag{12}$$

Since as we have said we are not concerned with complete generality, we will be satisfied in treating in order the following characteristic cases:

[9]cf. eg. E. Noether, Gött. Nach. 1918, p. 211. The divergence $\frac{d\mathbf{R}^\nu}{dx^\nu}$ appears if the integral $\int \mathcal{L} dx^1 \cdots dx^4$ is not invariant for an arbitrary integration domain but only when $\mathbf{R}^\nu$ vanishes on the boundary.

1. $\mathcal{L}' = 0$, i.e., $\mathcal{L}$ is itself a scalar density:

$$\delta\mathcal{L} + \mathcal{L}\frac{d\delta x^\nu}{dx^\nu} = 0; \tag{13}$$

2. $\mathcal{L}'$ contains second derivatives

$$Q_{\alpha,\nu\rho} \equiv \frac{\partial^2 Q_\alpha}{\partial x^\nu \partial x^\rho}$$

only linearly, i.e.,

$$\mathcal{L}' \equiv \frac{d}{dx^\nu}\underline{[f^{\nu,\alpha\rho}(Q)Q_{\alpha,\rho}]}, \tag{14}$$

and $j = 0$ (cf. equation (3)).

In both cases the investigation splits into two steps:
a) Implementation of the Hamiltonian method;
b) Proof of covariance of the Hamiltonian procedure under the relevant group.
We begin with the first case.

§2. The conjugate momentae and the identities

Henceforth we assume condition (13) is satisfied.

First we set

$$\mathcal{P}^{\alpha\nu} = \frac{\partial\mathcal{L}}{\partial Q_{\alpha,\nu}}, \tag{15}$$

and we take as momentae

$$\mathcal{P}^{\alpha} \equiv \mathcal{P}^{\alpha 4} = \frac{\partial\mathcal{L}}{\partial \dot{Q}_\alpha}. \tag{16}$$

We confine ourselves first to the classical c-number theory.

We substitute into (13) δQ_α, $Q_{\alpha,\nu}$ and δx^ν through their values given in (2) and (6) as functions of ξ^r and derivatives. We obtain several identities in expressing the fact that the coefficients of individual derivatives of ξ must identically vanish. These identities generally contain the $\dot{Q}_\alpha$ not only through the just introduced functions $\mathcal{P}^\alpha$ but also through other relations (e.g., through the other $\mathcal{P}^{\alpha\nu}, \nu \neq 4$); the system of equations (13) is therefore not of interest in solving for $\dot{Q}_\alpha$; it simply represents relations that each solution $\dot{Q}_\alpha(Q, \mathcal{P})$ must fulfill. If some of the identities under consideration contain only the Q_α (including spatial derivatives) and the $\mathcal{P}^\alpha$, then the relations are fundamentally different. They signify that the equations (2) are not independent so that the general solution will depend on arbitrary parameters (more precisely, spacetime functions).

The last case always occurs with the group (2). The highest derivative of ξ^ν in (13) is

$$\frac{\partial^{j+1}\xi^r}{\partial x^\sigma \cdots \partial x^\tau \partial x^\nu};$$

according to the assumption (3) the corresponding identities read

$$\sum \mathcal{P}^{\alpha\nu} c_{\alpha r}^{\sigma\cdots\tau} \equiv 0, \tag{17c}$$

[10]

where the summation runs over all permutations of the numbers $\nu, \sigma, \cdots, \tau$. For the case $\nu = \sigma = \cdots = \tau = 4$ one has in particular

$$\mathcal{P}^{\alpha} c_{\alpha r}^{44\cdots 4} \equiv 0; \tag{18c}$$

since the c contain only the Q_α we have in (18c) r_0 identities of the last type considered that we will call "proper" identities.[7] Furthermore it is easy to see that in general (i.e., in the case that the Lagrange function possesses no special properties) that no more identities appear. The general solution $\dot{Q}_\alpha(Q, \mathcal{P}, \lambda^0)$ of (16) depends on r_0 arbitrary parameters λ.

In the previous methods mentioned in the introduction one proceeded either through the destruction of the invariance properties of the Lagrangian (methods 1 and 3) or through the choice of a special solution $\dot{Q}_\alpha(Q, \mathcal{P}, \lambda^0)$ (method 2). In contrast the fundamental idea of the new method is to construct the Hamilton function using the general solution $\dot{Q}_\alpha(Q, \mathcal{P}, \lambda)$ with undetermined λ^r, without for the moment worrying about the proper identities. Field equations and commutation relations have the canonical form, with the former containing the λ^r. In this canonical formalism the proper identities ultimately become constraints. We will see that in addition to its simplicity the method has the advantage that the proof of covariance can be carried out without difficulty.

§3. Transition to q-numbers

In passing to q-numbers we must first investigate the form of the relations described above. According to (1) the relation (15) reads

$$\mathcal{P}^{\alpha\nu} = \frac{1}{2}(p^{\alpha\nu} + p^{\alpha\nu\dagger}) = \underline{p^{\alpha\nu}}, \tag{19}$$

with

$$p^{\alpha\nu} = \mathcal{A}^{\alpha\nu;\beta\mu} Q_{\beta,\mu} + \mathcal{B}^{\alpha\nu}. \tag{20}$$

A bar over an index of the form $\mu : \overline{\mu}$ signifies that the index ranges from 1 to 3; the summation convention will also hold for barred indices. With this notation according to (19) and (20) we write

$$\left\{ \begin{aligned} \mathcal{P}^\alpha &= \underline{p^\alpha}, \\ p^\alpha &= \mathcal{A}^{\alpha\beta}\dot{Q}_\beta + \mathcal{D}^\alpha, \end{aligned} \right. \tag{21}$$

[10]Equation numbers will be denoted with c when they have unlimited validity only for c-numbers.

where we define

$$(22)\qquad \begin{cases} \mathcal{A}^{\alpha\beta} \equiv \mathcal{A}^{\alpha 4;\beta 4}, \\ \mathcal{D}^{\alpha} \equiv \mathcal{A}^{\alpha 4;\beta\overline{\mu}} Q_{\beta,\overline{\mu}} + \mathcal{B}^{\alpha 4}. \end{cases}$$

We assume in this equation that

$$\mathcal{A}^{\alpha\nu;\beta\mu} = \mathcal{A}^{\beta\mu;\alpha\nu},$$

and in particular that

$$\mathcal{A}^{\alpha\beta} = \mathcal{A}^{\beta\alpha}$$

which is of course no additional restriction.

The considerations of the previous paragraph yield instead of (17c) and (18c)

$$(23)\qquad \sum \underline{c^{\sigma\cdots\tau}_{\alpha r} p^{\alpha\nu}} \equiv 0,$$

and

$$(24)\qquad \underline{c^{4\cdots 4}_{\alpha r} p^{\alpha}} \equiv 0.$$

Since in particular (24) holds identically in the $\dot{Q}_\alpha$ we have according to (21)

$$(25)\qquad c^{4\cdots 4}_{\alpha r} \mathcal{A}^{\alpha\beta} = 0,$$

$$(26)\qquad c^{4\cdots 4}_{\alpha r} \mathcal{D}^{\alpha} = 0;$$

the coefficients in the Lagrange function must satisfy these and other conditions in order that it possess that required density property. We display the relations (25) and (26) for later use.

We cannot proceed further without knowing something about the commutation relations $[Q_\alpha, \dot{Q}_\beta]$. If we were to know the $\dot{Q}_\beta$ as functions of the Q_α and $\mathcal{P}^\alpha$, then we would be able to derive the value of $[Q_\alpha, \dot{Q}_\beta]$ from the canonical commutation relations, which, as we have said, we wish to retain. But it is not clear a priori whether we can from (21) derive the $\dot{Q}_\beta$ as functions of the *matrices* $\mathcal{P}^\alpha$, or only as functions of the *matrix elements* of $\mathcal{P}^\alpha$. The only way we can overcome this problem is to make a tentative *assumption* about the $[Q_\alpha, \dot{Q}_\beta]$ on whose basis the solution of (21) takes the form $\dot{Q}_\alpha(Q, \mathcal{P}, \lambda)$. Later we can check whether the assumption is compatible with canonical commutation relations.

A related assumption is the following: the $[Q_\alpha, \dot{Q}_\beta]$ should be *anti-Hermitian* [11] functions of Q_α and $Q_{\alpha,\overline{\nu}}$, but not of the $\dot{Q}_\alpha$ (respectively the $\mathcal{P}^\alpha$).[8] (Whether undetermined factors like $\delta(0)$ appear when Q_α and $\dot{Q}_\beta$ are taken at the same location is irrelevant.) We now draw a few immediate conclusions from these assumptions:

1. According to (20) both $[Q_\alpha, p^{\beta\nu}]$ and $[Q_\alpha, p^{\beta\nu\dagger}]$ are anti-Hermitian functions of the Q_α and the $Q_{\alpha,\overline{\nu}}$ alone.

[11] A q-number x is said to be anti-Hermitian when $x^\dagger = -x$.

2. The $[Q_\alpha, \dot{Q}_\beta]$, $[Q_\alpha, p^{\beta\nu}]$ and $[Q_\alpha, p^{\beta\nu\dagger}]$ commute with every function of the Q_α and $Q_{\alpha,\overline{\nu}}$.

3. We have

$$[Q_\alpha, p^{\beta\nu}] = [Q_\alpha, p^{\beta\nu\dagger}]. \tag{27}$$

As a consequence instead of (23) and (24) we can write

$$\sum \underline{c_{\alpha r}^{\sigma\cdots\tau}\mathcal{P}^{\alpha\nu}} = 0, \tag{28}$$

$$\mathcal{F}_r \equiv \underline{c_{\alpha r}^{4\cdots 4}\mathcal{P}^{\alpha}} = 0. \tag{29}$$

From (25) it follows that the N linear equations

$$\mathcal{A}^{\alpha\beta}\dot{Q}_\beta + \dot{Q}_\beta\mathcal{A}^{\beta\alpha} = 2(\mathcal{P}^\alpha - \mathcal{D}^\alpha), \tag{21}$$

are not independent, rather, the determinant $|\mathcal{A}^{\alpha\beta}|$ has the rank $N - r_0$.[9] Since it is symmetric there exists a nonvanishing principal minor of degree $N - r_0$; we will denote the corresponding indices with a prime:

$$|\mathcal{A}^{\alpha'\beta'}| \neq 0,$$

while the remaining indices will be doubly primed: $\mu'', \nu'', \cdots$. The determinant $|\mathcal{A}^{\alpha'\beta'}|$ as well as its inverse $|\mathcal{A}_{\alpha'\beta'}|$ are symmetric, and we have

$$\mathcal{A}^{\alpha'\beta'}\mathcal{A}_{\beta'\gamma'} = \delta_{\gamma'}^{\alpha'}, \tag{30}$$

where as usual δ_γ^α equals 0 or 1 accordingly as $\alpha \neq \beta$ or $\alpha = \beta$.

If we succeed in finding a *special* solution $\dot{Q}_\beta^0(Q, \mathcal{P})$ of (21), then the general solution has the form:

$$\dot{Q}_\beta = \dot{Q}_\beta^0 + \lambda^r x_{\beta r},$$

where the λ^r are r_0 arbitrary parameters and the $x_{\beta r}$ represent r_0 independent solutions of the homogeneous equations

$$\mathcal{A}^{\alpha\beta}x_{\beta r} + x_{\beta r}\mathcal{A}^{\beta\alpha} = 0.$$

According to (25) we can now choose

$$x_{\beta r} = c_{\beta r}^{4\cdots 4}$$

and write

$$\dot{Q}_\beta = \dot{Q}_\beta^0 + \lambda^r c_{\beta r}^{4\cdots 4}, \tag{31}$$

Furthermore I maintain that

$$\begin{cases} \dot{Q}_{\beta'}^0 = \frac{1}{2}\left\{\mathcal{A}_{\beta'\gamma'}(\mathcal{P}^{\gamma'} - \mathcal{D}^{\gamma'}) + (\mathcal{P}^{\gamma'} - \mathcal{D}^{\gamma'})\mathcal{A}_{\gamma'\beta'}\right\} \\ \dot{Q}_{\beta''}^0 = 0. \end{cases} \tag{32}$$

is a special solution of (21). If this is verified, then we have actually succeeded in solving (21) for the $\dot{Q}_\beta$: the solution (31) manifestly has the required property since by virtue of the canonical commutation relations $[Q_\alpha, \dot{Q}_\beta]$ is an anti-Hermitian function of the Q_α and $Q_{\alpha,\overline{\mu}}$.

Substituting (32) into the left hand side of (21), [10] which for the moment we will call $\mathcal{T}_\alpha$ one obtains

$$\begin{aligned}
\mathcal{T}^\alpha &= \frac{1}{2}\mathcal{A}^{\alpha\beta'}\left\{\mathcal{A}_{\beta'\gamma'}(\mathcal{P}^{\gamma'}-\mathcal{D}^{\gamma'})+(\mathcal{P}^{\gamma'}-\mathcal{D}^{\gamma'})\mathcal{A}_{\gamma'\beta'}\right\}\\
&+ \frac{1}{2}\left\{\mathcal{A}_{\beta'\gamma'}(\mathcal{P}^{\gamma'}-\mathcal{D}^{\gamma'})+(\mathcal{P}^{\gamma'}-\mathcal{D}^{\gamma'})\mathcal{A}_{\gamma'\beta'}\right\}\mathcal{A}^{\beta'\alpha}\\
&= \mathcal{A}^{\alpha\beta'}\mathcal{A}_{\beta'\gamma'}(\mathcal{P}^{\gamma'}-\mathcal{D}^{\gamma'})+(\mathcal{P}^{\gamma'}-\mathcal{D}^{\gamma'})\mathcal{A}_{\gamma'\beta'}\mathcal{A}^{\beta'\alpha}\\
&+ \frac{1}{2}\mathcal{A}^{\alpha\beta'}[\mathcal{P}^{\gamma'}-\mathcal{D}^{\gamma'},\mathcal{A}_{\gamma'\beta'}]+\frac{1}{2}[\mathcal{A}_{\beta'\gamma'},\mathcal{P}^{\gamma'}-\mathcal{D}^{\gamma'}]\mathcal{A}^{\beta'\alpha}\\
&= \mathcal{A}^{\alpha\beta'}\mathcal{A}_{\beta'\gamma'}(\mathcal{P}^{\gamma'}-\mathcal{D}^{\gamma'})+(\mathcal{P}^{\gamma'}-\mathcal{D}^{\gamma'})\mathcal{A}_{\gamma'\beta'}\mathcal{A}^{\beta'\alpha}
\end{aligned}$$

by virtue of the second implication of our assumption.[11] For $\alpha = \alpha'$ we immediately deduce from (30)

$$\mathcal{T}^{\alpha'} = 2(\mathcal{P}^{\alpha'} - \mathcal{D}^{\alpha'}).$$

Now according to the theory of linear equations and using our assumption regarding $[Q_\alpha, \dot{Q}_\beta]$ the identities (29) are equivalent to

$$\mathcal{P}^{\alpha''} = \underline{\mathcal{A}^{\alpha''\beta'}\mathcal{A}_{\beta'\gamma'}\mathcal{P}^{\gamma'}},$$

and in the same manner (26) is equivalent to

$$\mathcal{D}^{\alpha''} = \mathcal{A}^{\alpha''\beta'}\mathcal{A}_{\beta'\gamma'}\mathcal{D}^{\gamma'}.$$

It follows that also

$$\mathcal{T}^{\alpha''} = 2(\mathcal{P}^{\alpha''} - \mathcal{D}^{\alpha''}),$$

whereby the proof is completed that (31), (32) represent the most general solution of (21) in agreement with the canonical commutation relations.[12]

§4. Construction of the Hamiltonian

Classically the Hamiltonian reads

$$\mathcal{H} = \mathcal{P}^\alpha \dot{Q}_\alpha - \mathcal{L};$$

in every quantum mechanical approach we must demand that

$$\frac{\partial \mathcal{H}}{\partial Q_{\alpha,\overline{\nu}}} = -\frac{\partial \mathcal{L}}{\partial Q_{\alpha,\overline{\nu}}}, \tag{33}$$

a property that will prove itself unavoidable in the elaboration of the theory.[13]

We have

$$\left(\frac{\partial\mathcal{L}}{\partial Q_{\alpha,\overline{\nu}}}\right)_{\mathcal{P}^\alpha} = \left(\frac{\partial\mathcal{L}}{\partial Q_{\alpha,\overline{\nu}}}\right)_{\dot{Q}_\alpha} + \underline{\left(\frac{\partial\dot{Q}_\beta}{\partial Q_{\alpha,\overline{\nu}}}\right)_{\mathcal{P}^\alpha} p^\beta}$$

and since according to (31 and (32), $\left(\frac{\partial\dot{Q}_\beta}{\partial Q_{\alpha,\overline{\nu}}}\right)_{\mathcal{P}^\alpha}$ does not contain the $\mathcal{P}^\alpha$ we can write

$$\left(\frac{\partial\mathcal{L}}{\partial Q_{\alpha,\overline{\nu}}}\right)_{\mathcal{P}^\alpha} = \left(\frac{\partial\mathcal{L}}{\partial Q_{\alpha,\overline{\nu}}}\right)_{\dot{Q}_\alpha} + \underline{\left(\frac{\partial\dot{Q}_\beta}{\partial Q_{\alpha,\overline{\nu}}}\right)_{\mathcal{P}^\alpha} \mathcal{P}^\beta}$$

The Ansatz

(34) $$\mathcal{H} = \underline{\dot{Q}_\alpha\mathcal{P}^\alpha} - \mathcal{L}.$$

therefore satisfies the condition (33). Since by (25) and (26)

$$\mathcal{L}[Q;\dot{Q}(Q,\mathcal{P},\lambda)] = \mathcal{L}(Q,\dot{Q}^0),$$

using the notation (29) we can write

(35) $$\mathcal{H} = \mathcal{H}_0 + \lambda^r\mathcal{F}_r,$$

with [14]

(36) $$\mathcal{H}_0 = \underline{\dot{Q}^0_\alpha\mathcal{P}^\alpha} - \mathcal{L}[Q,\dot{Q}^0(Q,\mathcal{P})].$$

Now we fix the canonical commutation relations

(37) $$\begin{cases} [Q_\alpha(\mathbf{r}),Q_\beta(\mathbf{r}')] = [\mathcal{P}^\alpha(\mathbf{r}),\mathcal{P}^\beta(\mathbf{r}')] = 0, \\ [\mathcal{P}^\alpha(\mathbf{r}),Q_\beta(\mathbf{r}')] = \omega\delta^\alpha_\beta\delta(\mathbf{r}-\mathbf{r}'),\ \omega = \frac{hc}{2\pi i}, \end{cases}$$

as well as the field equations [15]

$$[\overline{\mathcal{H}},Q_\alpha] = \omega\dot{Q}_\alpha,$$

(38) $$[\overline{\mathcal{H}},\mathcal{P}^\alpha] = \omega\dot{\mathcal{P}}^\alpha,$$

where we use the notation

(39) $$\overline{\mathcal{A}} \equiv \int \mathcal{A}dx^1dx^2dx^3;$$

the integration domain must be chosen in such a manner that field quantities assume a constant value on the boundary, indeed, such values that $\mathcal{L}$ vanishes there. [16]

In addition to (37) and (38) we have the proper identities (29) $\mathcal{F}_r = 0$ as constraints. But it must be proven that it is permissible to set all of the $\mathcal{F}_r$

simultaneously to zero; in other words, that the $\mathcal{F}_r$ commute with each other, at least on account of the constraints $\mathcal{F}_r = 0$ themselves.

The following observations will serve not only this purpose, but are also the basis for the proof of covariance to be adduced later on.

We define first the energy-momentum pseudo tensor [12]

$$\mathcal{G}^{\nu}_{\mu} = \underline{\mathcal{P}^{\alpha\nu} Q_{\alpha,\mu}} - \delta^{\nu}_{\mu}\mathcal{L}, \tag{40}$$

and then energy-momentum pseudo density

$$\mathcal{G}_{\mu} = \mathcal{G}^{4}_{\mu} = \underline{\mathcal{P}^{\alpha} Q_{\alpha,\mu}} - \delta^{4}_{\mu}\mathcal{L}, \tag{41}$$

whose fourth pseudo component is the Hamiltonian function (34):

$$\mathcal{H} = \mathcal{G}_4 = \mathcal{G}^4_4.$$

The components of the total momentum are then $\overline{\mathcal{G}_{\overline{\nu}}}$ and the total energy is $\overline{\mathcal{H}}$.

The CR (commutation relations) of $\overline{\mathcal{H}}$ with the $Q_\alpha, \mathcal{P}^\alpha$ are given by (38). Concerning the $\overline{\mathcal{G}_{\overline{\mu}}}$ we first find referring to (37) that

$$\left\{ \begin{array}{l} [\mathcal{G}_{\overline{\nu}}(\mathbf{r}), Q_\alpha(\mathbf{r}')] = \omega \frac{\partial Q_\alpha}{\partial x^{\overline{\nu}}} \delta(\mathbf{r}-\mathbf{r}'), \\ [\mathcal{G}_{\overline{\nu}}(\mathbf{r}), \mathcal{P}^\alpha(\mathbf{r}')] = -\omega \mathcal{P}^\alpha \frac{\partial \delta(\mathbf{r}-\mathbf{r}')}{\partial x^{\overline{\nu}}}, \end{array} \right. \tag{42}$$

so that

$$[\overline{\mathcal{G}_{\overline{\nu}}}, \Phi(Q, \mathcal{P})] = \omega \frac{d\Phi}{dx^{\overline{\nu}}},$$

and therefore more generally,

$$\omega \frac{d\Phi}{dx^{\overline{\nu}}} = [\overline{\mathcal{G}_{\overline{\mu}}}, \Phi(Q, \mathcal{P}, x)] + \omega \frac{\partial \Phi}{\partial x^{\overline{\nu}}}, \tag{43}$$

from which it follows immediately that [17]

$$[\overline{\mathcal{G}_\mu}, \overline{\mathcal{G}_\nu}] = 0: \tag{44}$$

this constitutes an expression for the commutability of the derivatives $\frac{d}{dx^\nu}$ whose physical content is the constancy in time of the $\overline{\mathcal{G}_\mu}$ that follows from equations (38) , (37). [13]

§5. Quantum-mechanical expression of the infinitesimal transformation group

In this paragraph we prove the the proposition: [18]

$$\omega \delta^* \Phi(Q, \mathcal{P}) = [\overline{\mathcal{M}}, \Phi], \tag{45}$$

[12] The prefix "pseudo" signifies that the relevant quantity is not a tensor.

[13] In case the λ^r contain the x^4 explicitly, (44) holds based on the constraints (29).

where

$$\mathcal{M} = \underline{\mathcal{P}^\alpha \delta Q_\alpha} - \mathcal{G}_\mu \delta x^\mu. \tag{46}$$

This should hold on account of the field equations (38) and the CR (31), under the assumption that $\mathcal{L}$ is a scalar density.

To prove this proposition it will suffice to show that

$$\left\{ \begin{array}{l} \omega\delta^* Q_\alpha = [\overline{\mathcal{M}}, Q_\alpha], \\ \omega\delta^* \mathcal{P}^\alpha = [\overline{\mathcal{M}}, \mathcal{P}^\alpha]. \end{array} \right. \tag{47}$$

According to (37) and (42), considering that by (2) δQ_α contains only the Q_α (and not the $\mathcal{P}^\alpha$),

$$[\overline{\mathcal{M}}, Q_\alpha] = \omega\delta Q_\alpha - \frac{dQ_\alpha}{dx^{\overline{\mu}}}\delta x^{\overline{\mu}} - [\overline{\delta x^4 \mathcal{H}}, Q_\alpha].$$

Now according to H. P. I, equation (20),

$$\left\{ \begin{array}{c} [\overline{\delta x^4 \mathcal{H}}, Q_\alpha] = \omega \frac{\partial(\delta x^4 \mathcal{H})}{\partial \mathcal{P}^\alpha} = \delta x^4 [\overline{\mathcal{H}}, Q_\alpha], \\ [\overline{\delta x^4 \mathcal{H}}, \mathcal{P}^\alpha] = -\omega \left\{ \frac{\partial(\delta x^4 \mathcal{H})}{\partial Q_\alpha} - \frac{d}{dx^{\overline{\nu}}} \frac{\partial(\delta x^4 \mathcal{H})}{\partial Q_{\alpha,\overline{\nu}}} \right\} \\ = \delta x^4 [\overline{\mathcal{H}}, \mathcal{P}^\alpha] + \omega \frac{d\delta x^4}{dx^{\overline{\nu}}} \frac{\partial \mathcal{H}}{\partial Q_{\alpha,\overline{\nu}}}. \end{array} \right. \tag{48}$$

The first relation (47) therefore follows referring to (4), using the first field equation (38).

Similarly, referring to the second field equation (38) and equation (33) [19]

$$\left\{ \begin{array}{c} \frac{1}{\omega}[\overline{\mathcal{M}}, \mathcal{P}^\alpha] = \underline{-\mathcal{P}^\beta \frac{\partial \delta Q_\beta}{\partial Q_\alpha}} - \frac{d}{dx^{\overline{\nu}}}(\mathcal{P}^\alpha \delta x^{\overline{\nu}}) \\ -\frac{d\mathcal{P}^\alpha}{dx^4}\delta x^4 + \mathcal{P}^{\alpha\overline{\nu}} \frac{d\delta x^4}{dx^{\overline{\nu}}}, \\ = \underline{-\mathcal{P}^\beta \frac{\partial \delta Q_\beta}{\partial Q_\alpha}} + \mathcal{P}^{\alpha\nu} \frac{d\delta x^4}{dx^\nu} - \frac{d}{dx^\nu}(\mathcal{P}^\alpha \delta x^\nu). \end{array} \right. \tag{49}$$

It remains only to show that the right hand side of (49) is equal to $\delta^* \mathcal{P}^\alpha$. We therefore calculate $\delta \mathcal{P}^\alpha$ directly, or more generally $\delta \mathcal{P}^{\alpha\nu}$. First we have [20]

$$\delta \mathcal{P}^{\alpha\nu} = \delta \left(\frac{\partial \mathcal{L}}{\partial Q_{\alpha,\nu}} \right) = \frac{\partial(\delta \mathcal{L})}{\partial Q_{\alpha,\nu}} - \underline{\frac{\partial \mathcal{L}}{\partial Q_{\beta,\mu}} \frac{\partial \delta Q_{\beta,\mu}}{\partial Q_{\alpha,\nu}}}, \tag{50}$$

for general c-numbers, and for q-numbers whenever $\mathcal{L}$ has the form (1) and $\frac{\partial \delta Q_{\beta,\mu}}{\partial Q_{\alpha,\nu}}$ does not contain the $\dot{Q}_\alpha$ (respectively, the $\mathcal{P}^\alpha$). The latter is true in our case according to the formula (6) that gives

$$\begin{aligned} \frac{\partial}{\partial Q_{\alpha,\nu}} \delta Q_{\beta,\mu} &= \frac{\partial}{\partial Q_{\alpha,\nu}} \left\{ \frac{d}{dx^\mu} \delta Q_\beta - Q_{\beta,\rho} \frac{d\delta x^\rho}{dx^\mu} \right\} \\ &= \frac{\partial \delta Q_\beta}{\partial Q_\alpha} \delta^\nu_\mu - \frac{d\delta x^\nu}{dx^\mu} \delta^\alpha_\beta. \end{aligned}$$

Substituting this into (50) yields

$$\delta\mathcal{P}^{\alpha\nu} = \underline{-\mathcal{P}^{\beta\nu}\frac{\partial\delta Q_\beta}{\partial Q_\alpha}} + \mathcal{P}^{\alpha\mu}\frac{d\delta x^\nu}{dx^\mu} + \frac{\partial(\delta\mathcal{L})}{\partial Q_{\alpha,\nu}};$$

now using (13) we have

$$\delta\mathcal{P}^{\alpha\nu} = \underline{-\mathcal{P}^{\beta\nu}\frac{\partial\delta Q_\beta}{\partial Q_\alpha}} + \mathcal{P}^{\alpha\mu}\frac{d\delta x^\nu}{dx^\mu} - \mathcal{P}^{\alpha\nu}\frac{d\delta x^\mu}{dx^\mu}; \tag{51}$$

i.e., as the comparison with (11) instructs us: $\mathcal{P}^{\alpha\nu}$ is a tensor density. From (51) with reference to (4) the expression (49) follows immediately for $\delta^*\mathcal{P}^\alpha \equiv \delta^*\mathcal{P}^{\alpha 4}$. The formula (45) is hereby proven.

§6. The $\overline{\mathcal{F}_r}$ as special infinitesimal transformations

We consider a fixed but arbitrary slice $x^4 = x^4_0$. On this slice we consider the transformations of our group (2) that are defined through the conditions

$$\begin{cases} (\xi^r)_{x^4=x^4_0} = \left(\frac{\partial\xi^r}{\partial x^\sigma}\right)_{x^4=x^4_0} = \cdots = \left(\frac{\partial^{j-1}\xi^r}{\partial x^\sigma\cdots\partial x^\tau}\right)_{x^4=x^4_0} = 0, \\ \left(\frac{\partial^j\xi^r}{\partial x^\sigma\cdots\partial x^\tau}\right)_{x^4=x^4_0} = 0, \text{when all of the } \sigma\cdots\tau \text{ are not equal to } 4, \\ \left(\frac{\partial^j\xi^r}{(\partial x^4)^j}\right)_{x^4=x^4_0} = \epsilon^r, \end{cases} \tag{52}$$

where the ϵ^r are arbitrary spatial functions.

On account of the assumption (3) these transformations do not lead out of the $x^4 = x^4_0$ slice. They constitute at every point in the slice a finite continuous subgroup of the group (2), whose infinitesimal transformations are given by [21]

$$\omega\delta'\Phi(Q,\mathcal{P}) = [\overline{\epsilon^r\mathcal{F}_r}, \Phi].$$

(The $Q, \mathcal{P}, \mathcal{F}$ are hereby taken at $x^4 = x^4_0$.)

The second fundamental proposition of Lie on finite transformation groups declares when applied to this subgroup that at every point of the slice

$$[\mathcal{F}_r, [\mathcal{F}_s, \Phi]] - [\mathcal{F}_s, [\mathcal{F}_r, \Phi]] = c^t_{rs}[\mathcal{F}_t, \Phi],$$

where the c^t_{rs} are the point (x^1, x^2, x^3, x^4_0) dependent group "structure constants". By the Jacobi bracket identity the left hand side is simply equal to

$$[[\mathcal{F}_s, \mathcal{F}_r], \Phi]\,;$$

we therefore obtain [22]

$$[\mathcal{F}_s, \mathcal{F}_r] = c^t_{rs}\mathcal{F}_t. \tag{53}$$

From here follows an additional fact required in the establishment of the method presented in section 4. Since $\mathcal{F}_r = 0$ the $\mathcal{F}_r$ must commute with each other.

§7. The infinitesimal transformations $\overline{\mathcal{M}}$ as integrals of the motion

Let us return for the moment to the pure c-number theory. We set

$$\mathcal{M}^\nu = \underline{\mathcal{P}^{\alpha\nu}\delta Q_\alpha} - \mathcal{G}^\nu_\mu \delta x^\mu, \tag{54}$$

and

$$\mathcal{L}^\alpha = \frac{\partial \mathcal{L}}{\partial Q_\alpha} - \frac{d}{dx^\nu}\frac{\partial \mathcal{L}}{\partial Q_{\alpha,\nu}}, \tag{55}$$

so as is easy to see that the assumption (13) is equivalent to [23]

$$\frac{d\mathcal{M}^\nu}{dx^\nu} + \mathcal{L}^\alpha \delta^* Q_\alpha = 0; \tag{56c}$$

taking into account that according to (46) and (54)

$$\mathcal{M} \equiv \mathcal{M}^4$$

and using the notation (39) we have [24]

$$\overline{\frac{d\mathcal{M}^{\overline{\nu}}}{dx^{\overline{\nu}}}} = 0,$$

then it follows from (56c) that

$$\frac{d\overline{\mathcal{M}}}{dx^4} = -\overline{\mathcal{L}^\alpha \delta^* Q_\alpha}. \tag{57c}$$

Now it is well known that the Hamiltonian equations (38) (by virtue of the proper identities (29)) are equivalent to the Lagrangian equations

$$\mathcal{L}^\alpha = 0.$$

Consequently by (57c), based on (13) and (38),

$$\frac{d\overline{\mathcal{M}}}{dx^4} = 0. \tag{58}$$

Equation (56c) cannot be carried over to q-numbers. Nevertheless the derivation of (56c) succeeds through use of the same assumptions (13) and (38), only under somewhat different circumstances. Both relations (13) and (38) were essential in the derivation of (43) and (45). Let us apply these latter results to the identity (5), whereby we let Φ depend only on Q and $\mathcal{P}$: [25]

$$[\overline{\mathcal{M}}, [\overline{\mathcal{G}}_\nu, \Phi]] = [\overline{\mathcal{G}}_\nu, [\overline{\mathcal{M}}, \Phi]] + \left[\omega \frac{\partial \overline{\mathcal{M}}}{\partial x^\nu}, \Phi\right],$$

$$\left[[\overline{\mathcal{G}}_\nu, \overline{\mathcal{M}}] + \omega \frac{\partial \overline{\mathcal{M}}}{\partial x^\nu}, \Phi\right] = 0$$

using the Jacobian identity, or finally

$$\left[\frac{d\overline{\mathcal{M}}}{dx^\nu}, \Phi\right] = 0.$$

In particular $\frac{d\overline{\mathcal{M}}}{dx^4}$ is c-number (generally dependent on x^4). But this c-number, as a sum of manifest q-numbers, can be nothing other than zero. This conclusion is confirmed through a careful calculation of $\frac{d\overline{\mathcal{M}}}{dx^4}$.

Interesting conclusions regarding the relation between $\overline{\mathcal{M}}$ and the functions $\mathcal{F}_r$ can be drawn from (58). Under integration by parts $\overline{\mathcal{M}}$ takes the form

$$\overline{\mathcal{M}} = \int dx^1 dx^2 dx^3 \sum_{i=0}^{i=j} \mathcal{N}_r{}^i \frac{\partial^i \xi^r}{(\partial x^4)^i}, \tag{59}$$

where

$$\mathcal{N}_r{}^j \equiv \mathcal{F}_r. \tag{60}$$

Equation (58) is then written as follows:

$$\int dx^1 dx^2 dx^3 \sum_{i=0}^{i=j} \mathcal{N}_r{}^i \frac{\partial^{i+1} \xi^r}{(\partial x^4)^{i+1}} = -\int dx^1 dx^2 dx^3 \sum_{i=0}^{i=j} \frac{d\mathcal{N}_r{}^i}{dx^4} \frac{\partial^i \xi^r}{(\partial x^4)^i}.$$

It follows through comparison of coefficients that

$$\mathcal{N}_r{}^i = -\frac{d\mathcal{N}_r{}^{i+1}}{dx^4} \; (i = 0, 1, \cdots, j-1), \tag{61}$$

and

$$\mathcal{N}_r{}^j = 0, \quad \frac{d\mathcal{N}_r{}^0}{dx^4} = 0. \tag{62}$$

(60) and (61) then yield

$$\mathcal{N}_r{}^i = (-1)^{j-i} \frac{d^{j-i}\mathcal{F}_r}{(dx^4)^{j-i}} \; (i = 0, 1, \cdots, j), \tag{63}$$

and $\overline{\mathcal{M}}$ therefore assumes the remarkable form

$$\overline{\mathcal{M}} = \int dx^1 dx^2 dx^3 \sum_{i=0}^{i=j} (-1)^{j-i} \frac{d^{j-i}\mathcal{F}_r}{(dx^4)^{j-i}} \frac{\partial^i \xi^r}{(\partial x^4)^i}. \tag{63$'$}$$

The first identity (62) is according (60) trivially $\mathcal{F}_r = 0$, The second shows, however, that on account of the field equations and the identities (29)

$$\frac{d^{j+1}\mathcal{F}_r}{(dx^4)^{j+1}} = 0. \tag{64}$$

This provides the answer to the question to what extent differentiation of the constraints (29) yields new constraints. [26]

If in particular $j = 1$, then the only new equations are $\frac{d\mathcal{F}_r}{dx^4} = 0$, i.e.,

$$[\overline{\mathcal{H}}, \mathcal{F}_r] + \omega \frac{\partial \mathcal{F}_r}{\partial x^4} = 0.$$

If the Lagrangian is of the form (1), i.e., if (35) holds, then referring to (53) this last equation becomes

$$[\overline{\mathcal{H}_0}, \mathcal{F}_r] + \omega \frac{\partial \mathcal{F}_r}{\partial x^4} + c^t_{rs} \lambda^s \mathcal{F}_t = 0,$$

or since $\mathcal{F}_r = 0$

$$[\overline{\mathcal{H}_0}, \mathcal{F}_r] + \omega \frac{\partial \mathcal{F}_r}{\partial x^4} = 0;$$

since neither the constraints or the new equations contain the λ^r they remain essentially undetermined. (However, in case $j > 1$ then already the $\frac{d^2\mathcal{F}_r}{(dx^4)^2}$ contain the λ^r.) [27] As a consequence of the essential indeterminateness of the λ^r, r_0 field equations of the form

$$\omega \dot{\mathcal{P}}^\alpha = [\overline{\mathcal{H}}, \mathcal{P}^\alpha]$$

are missing. [28] The equations

$$\mathcal{N}_r{}^0 \equiv \frac{d\mathcal{F}_r}{dx^4} = 0, \text{ i.e. } [\overline{\mathcal{H}_0}, \mathcal{F}_r] + \omega \frac{\partial \mathcal{F}_r}{\partial x^4} = 0$$

just suffice as a replacement. [29]

In the case $j = 0$ the missing field equations are replaced by the identities $\mathcal{F}_r = 0$ themselves, which according to (64), i.e., $\frac{d\mathcal{F}_r}{dx^4} = 0$, evolve in time.

We want to make one last observation with regard to the formula (63′). We inquire into the subgroup of our group that leaves all of the points of the slice $x^4 = x^4_0$ invariant; this group is manifestly a normal subgroup. The conditions

$$\delta x^\nu = 0 \text{ for } x^4 = x^4_0$$

imply that

$$(\xi^r)_{x^4=x^4_0} = \left(\frac{\partial \xi^r}{\partial x^\sigma}\right)_{x^4=x^4_0} = \cdots = \left[\frac{\partial^k \xi^r}{\partial x^\sigma \cdots \partial x^\tau}\right]_{x^4=x^4_0} = 0;$$

the infinitesimal transformation then reads, referring to (63′)

$$\overline{\mathcal{S}} = \int dx^1 dx^2 dx^3 \sum_{i=k+1}^{i=j} (-1)^{j-i} \frac{d^{j-i}\mathcal{F}_r}{(dx^4)^{j-i}} s_i{}^r, \tag{65}$$

where $\frac{d^{j-i}\mathcal{F}_r}{(dx^4)^{j-i}}$ is taken at $x^4 = x^4_0$ and the

$$s_i{}^r \equiv \left[\frac{\partial^i \xi^r}{(\partial x^4)^i}\right]_{x^4=x^4_0}$$

are arbitrary spatial functions.[30] The group $\overline{S}$ is at every point of the slice a $r_0(j-k)$parametric invariant subgroup. The group (52) considered in section 6 is a subgroup of this group.

§8. Covariance of the procedure under the group action

Using the results just obtained we are now in position to easily settle the question whether the procedure is covariant.

The formula (45) implies that for an arbitrary group transformation every functional $\Phi(Q,\mathcal{P})$ is subject to a similarity transformation of the form

$$\Phi' = S^{-1}\Phi S \tag{66}$$

where according to (58) S is time independent.

Furthermore, as is easy to see [14] that formula (45) is also true for infinitesimal transformations $\overline{\mathcal{N}}$ of the group, i.e., when all of the field quantities are subject to the infinitesimal transformation $\overline{\mathcal{M}}$,

$$\omega\delta^*\overline{\mathcal{N}} = [\overline{\mathcal{M}},\overline{\mathcal{N}}], \tag{67}$$

hence, more generally,

$$\overline{\mathcal{N}}' = S^{-1}\overline{\mathcal{N}}S. \tag{67$'$}$$

From (66) the covariance of the CR (37) follows immediately. According to (35) the Hamiltonian consists of a Q and $\mathcal{P}$ dependent functional $\overline{\mathcal{H}_0}$ and a term $\overline{\lambda^r\mathcal{F}_r}$ that according to section 6 represents a special infinitesimal transformation $\overline{\mathcal{N}}$. On account of (66) and (67′) the canonical field equations are also subject to a (constant in time) unitary transformation, under which, as is well known, they remain invariant.

All that remains to investigate is the variation of the left hand sides $\mathcal{F}_r$ of the identities (29). According to (67) they vary as

$$\omega\delta^*\mathcal{F}_r = [\overline{\mathcal{M}},\mathcal{F}_r]. \tag{68}$$

[14] If one replaces Φ in

$$\Phi' = \Phi + \frac{1}{\omega}[\overline{\mathcal{N}},\Phi]$$

by

$$\tilde{\Phi} = \Phi + \frac{1}{\omega}[\overline{\mathcal{M}},\Phi]$$

and Φ' by

$$\tilde{\Phi}' = \Phi' + \frac{1}{\omega}[\overline{\mathcal{M}},\Phi']$$

then it follows after an easy calculation that

$$\tilde{\Phi}' = \tilde{\Phi} + \frac{1}{\omega}\left[\overline{\mathcal{N}} + \frac{1}{\omega}[\overline{\mathcal{M}},\overline{\mathcal{N}}],\tilde{\Phi}\right],$$

cf. also E. Noether, Gött. Nach. 1918, p. 252. [31]

Thus it follows from the fact that the group $\overline{\mathcal{S}}$ defined in (65) is an invariant subgroup, [32]

$$[\overline{\mathcal{M}}, \mathcal{F}_r] = \sum_{i=k+1}^{i=j} \alpha_i^{rs} \frac{d^{j-i}\mathcal{F}_s}{(dx^4)^{j-i}}. \tag{68'}$$

According to (68) and (68′) we therefore have $\delta^* \mathcal{F}_r = 0$, i.e., the proper identities $\mathcal{F}_r = 0$ are invariant, due indeed to the identities themselves and possible time derivatives thereof.

§9. Extension of the theory to the "second case" of section 1

We indicate briefly how the theory above is extended to the "second case" defined in section 1.

Our group would then have the simple form:

$$\delta x^\nu = 0,$$

$$\delta Q_\alpha = c_{\alpha r}\xi^r. \tag{69}$$

With

$$\mathcal{L}' \equiv \frac{d}{dx^\nu}[\underline{f^{\nu,\alpha\rho}(Q)Q_{\alpha,\rho}}] \tag{14}$$

we have according to (12)

$$\delta(\mathcal{L} + \mathcal{L}') = 0. \tag{70}$$

1. Next we calculate $\delta\mathcal{L}'$. I maintain that $\delta\mathcal{L}'$ takes the form

$$\delta\mathcal{L}' = \frac{d}{dx^\nu}\left(\underline{\mathcal{R}^{\alpha\nu}\delta Q_\alpha}\right) \tag{71}$$

or

$$\delta\mathcal{L}' = \frac{d}{dx^\nu}\left(\mathcal{I}_r^\nu \xi^r\right). \tag{72}$$

We obtain first

$$\delta\mathcal{L}' = \frac{d}{dx^\nu}\left\{\underline{\frac{\partial f^{\nu,\alpha\rho}}{\partial Q_\beta} c_{\beta r}\xi^r Q_{\alpha,\rho} + f^{\nu,\alpha\rho}\frac{d(c_{\alpha r}\xi^r)}{dx^\rho}}\right\};$$

we set

$$r^{\alpha\nu} = -\frac{df^{\nu,\alpha\rho}}{dx^\rho} + Q_{\beta,\rho}\frac{\partial f^{\nu,\beta\rho}}{\partial Q_\alpha}, \tag{73}$$

and

$$\mathcal{I}_r{}^\nu = \underline{r^{\alpha\nu} c_{\alpha r}}, \tag{74}$$

so we have

$$\delta\mathcal{L}' = \frac{d}{dx^\nu}\left\{\mathcal{I}_r{}^\nu \xi^r + \frac{d}{dx^\rho}(f^{\nu,\alpha\rho} c_{\alpha r} \xi^r)\right\}. \tag{75}$$

Now we use (70) and write out the requirement that the coefficients of the second derivatives of the ξ^r vanish identically. Since $\mathcal{L}$ contains no second derivatives of the ξ^r, we have according to (75)

$$(f^{\nu,\alpha\rho} + f^{\rho,\alpha\nu}) c_{\alpha r} \equiv 0. \tag{76}$$

As a consequence (75) is reduced to

$$\delta\mathcal{L}' = \frac{d}{dx^\nu}(\mathcal{I}_r{}^\nu \xi^r).$$

Now we set

$$\mathcal{R}^{\alpha\nu} = \underline{r^{\alpha\nu}} \tag{77}$$

and notice that instead of (74) we can also write

$$\mathcal{I}_r{}^\nu = \underline{\mathcal{R}^{\alpha\nu} c_{\alpha r}}, \tag{74}$$

thus we have proven formulas (71) and (72).

2. Now we set up the analogues of the identities (28) that in the first case contain the proper identities (29). For that purpose we must simply set the coefficients of the $\frac{d\xi^r}{dx^\nu}$ in (70) equal to zero. We obtain

$$\underline{(\mathcal{P}^{\alpha\nu} + \mathcal{R}^{\alpha\nu}) c_{\alpha r}} = 0. \tag{78}$$

In particular for $\nu = 4$:

$$\underline{(\mathcal{P}^{\alpha} + \mathcal{R}^{\alpha 4}) c_{\alpha r}} = 0,$$

or, once again substituting

$$\mathcal{F}_r = \underline{\mathcal{P}^\alpha c_{\alpha r}} \text{ and } \mathcal{I}_r{}^4 = \mathcal{I}_r,$$

$$\mathcal{F}_r + \mathcal{I}_r = 0. \tag{79}$$

3. The identities (79) are proper, i.e., we have

$$\frac{\partial \mathcal{I}_r}{\partial \dot{Q}_\alpha} = 0. \tag{80}$$

More generally we wish to prove that instead of (80)

$$\frac{\partial(r^{\beta\rho}c_{\beta r})}{\partial Q_{\alpha,\nu}} = -\frac{\partial(r^{\beta\nu}c_{\beta r})}{\partial Q_{\alpha,\rho}}, \tag{80$'$}$$

from which (80) follows from (74) when $\nu = \rho = 4$.

For this purpose we set equal to zero the coefficients of ξ^r in (70): there is one in the term linear in the second derivative $Q_{\alpha,\rho\nu}$, with only Q dependent coefficients. Since this expression vanishes for arbitrary $Q_{\alpha,\rho\nu}$, we can in particular assign c-number values to the $Q_{\alpha,\rho\nu}$ and then separately set to zero the coefficients of the $Q_{\alpha,\rho\nu}$. Using the formula that is valid for arbitrary $\mathcal{K}^\rho(Q_\alpha; Q_{\alpha,\nu})$:

$$\frac{\partial}{\partial Q_{\alpha,\nu}}\frac{d}{dx^\rho}\mathcal{K}^\rho(Q_\alpha; Q_{\alpha,\nu}) = \frac{\partial \mathcal{K}^\nu}{\partial Q_\alpha} + \frac{d}{dx^\rho}\frac{\partial \mathcal{K}^\rho}{\partial Q_{\alpha,\nu}}, \tag{81}$$

we find for these coefficients according to (71 and (73) [33]

$$\frac{\partial(r^{\beta\rho}c_{\beta r})}{\partial Q_{\alpha,\nu}} + \frac{\partial(r^{\beta\nu}c_{\beta r})}{\partial Q_{\alpha,\rho}};$$

setting this expression equal to zero gives (80$'$).

According to (81) it follows furthermore from (73) that [34]

$$\frac{\partial r^{\alpha\nu}}{\partial Q_{\beta,\rho}} = \frac{\partial f^{\nu,\beta\rho}}{\partial Q_\alpha} - \frac{\partial f^{\nu,\alpha\rho}}{\partial Q_\beta} = \frac{\partial \mathcal{R}^{\alpha\nu}}{\partial Q_{\beta,\rho}}; \tag{82}$$

therefore instead of (80$'$) we can write

$$\frac{\partial(\mathcal{R}^{\beta\rho}c_{\beta r})}{\partial Q_{\alpha,\nu}} = -\frac{\partial(\mathcal{R}^{\beta\nu}c_{\beta r})}{\partial Q_{\alpha,\rho}}. \tag{83}$$

4. The calculations of sections 3 and 4 can be applied word for word to the present case with $\mathcal{P}^\alpha + \mathcal{R}^{\alpha 4}$ taking the role of $\mathcal{P}^\alpha$.

The derived expression $\overline{\mathcal{M}}$ for infinitesimal transformations undergoes an analogous modification since $\mathcal{P}^{\alpha\nu}$ is no longer a tensor density. [15]

Rather, we have according to (50) and (70)

$$\delta\mathcal{P}^{\alpha\nu} = -\mathcal{P}^{\beta\nu}\frac{\partial\delta Q_\beta}{\partial Q_\alpha} - \frac{\partial(\delta\mathcal{L}')}{\partial Q_{\alpha,\nu}};$$

but by (71), referring to (81),

$$\frac{\partial(\delta\mathcal{L}')}{\partial Q_{\alpha,\nu}} = \frac{\partial(\mathcal{R}^{\beta\nu}\delta Q_\beta)}{\partial Q_\alpha} + \frac{d}{dx^\rho}\frac{\partial(\mathcal{R}^{\beta\rho}\delta Q_\beta)}{\partial Q_{\alpha,\nu}},$$

[15] Although neither $\mathcal{P}^{\alpha\nu}$ nor $\mathcal{R}^{\alpha\nu}$ are tensor densities, it is easy to show that $\mathcal{P}^{\alpha\nu} + \mathcal{R}^{\alpha\nu}$ is a tensor density.

i.e., taking (83) into account

$$\delta\mathcal{P}^{\alpha\nu} = \underline{-\mathcal{P}^{\beta\nu}\frac{\partial\delta Q_\beta}{\partial Q_\alpha} - \frac{\partial(\mathcal{R}^{\beta\nu}\delta Q_\beta)}{\partial Q_\alpha} + \frac{d}{dx^\rho}\frac{\partial(\mathcal{R}^{\beta\nu}\delta Q_\beta)}{\partial Q_{\alpha,\rho}}}. \tag{84}$$

In particular due to (80), for $\nu = 4$ this becomes [35]

$$\delta\mathcal{P}^{\alpha} = \underline{-\mathcal{P}^{\beta}\frac{\partial\delta Q_\beta}{\partial Q_\alpha} - \frac{\partial(\mathcal{R}^{\beta 4}\delta Q_\beta)}{\partial Q_\alpha} + \frac{d}{dx^{\overline{\rho}}}\frac{\partial(\mathcal{R}^{\beta 4}\delta Q_\beta)}{\partial Q_{\alpha,\overline{\rho}}}},$$

i.e.,

$$\omega\delta\mathcal{P}^{\alpha} = [\overline{\mathcal{N}}, \mathcal{P}^{\alpha}], \tag{85}$$

with

$$\mathcal{N} = \underline{(\mathcal{P}^{\alpha} + \mathcal{R}^{\alpha 4})\delta Q_\alpha}. \tag{86}$$

Once again on account of (80) we also have [36]

$$\omega\delta Q_\alpha = [\overline{\mathcal{N}}, Q_\alpha], \tag{87}$$

so that we have in $\overline{\mathcal{N}}$ the desired extension of $\overline{\mathcal{M}}$.

From expression (86) it follows exactly as in section 6 that the left hand sides $\mathcal{F}_r + \mathcal{I}_r$ of the proper identities commute as a consequence of the identities themselves.

The considerations of section 7 concerning the constancy in time of the $\overline{\mathcal{M}}$ as well as the proof of covariance of section 8 can be carried over without change to $\overline{\mathcal{N}}$. In particular, since it is assumed here that $j = 0$, the identities $\mathcal{F}_r + \mathcal{I}_r = 0$ play the role of the missing field equations.

§10. Observations concerning the simultaneous treatment of multiple groups

In case the Lagrangian admits several groups the above theory is still applicable considering that the infinitesimal transformation of the direct product of the relevant groups consists of the sum of infinitesimal transformations of the individual groups. In particular the $\mathcal{F}_r$ of each individual group commute not only with each other (due to $\mathcal{F}_r = 0$), but also with the $\mathcal{F}_r$ belonging to other groups. It is also permissible that "case 1" ($\mathcal{L}$ is a density) may apply to some groups, and "case 2" treated in section 9 may apply to others. For the latter case we must simply replace the $\mathcal{F}_r$ by $\mathcal{F}_r + \mathcal{I}_r$; these once again commute not only among themselves but also with the remaining $\mathcal{F}_r$.

It follows from this observation that one may treat independently the individual groups admitted by the Lagrangian.

Part Two: Applications

§11. The Lagrangian

We wish to construct a Lagrangian that includes not only electromagnetic and material fields, but also the gravitational field. Concerning the latter, we will adopt the one-body theory proposed by Fock [16] and Weyl [17]: we describe the gravitational field through the introduction at every point of four orthogonal vectors $h_{i,\nu}, (i = 1,2,3,4)$ and we demand that the laws of nature be covariant under x-dependent Lorentz transformations of the "Vierbeine" . [37] This covariance, called "Beincovariance" by Levi Civita [18] is fundamentally different from the "local Bein covariance" demanded by the Einsteinian theory of Fernparallelismus in which all of the tetrads are rigidly linked (*constant* Lorentz transformations of the tetrads). In agreement with Fock (and contrary to Weyl) we describe the material field through a *four*-component wave function $\psi \equiv (\psi_1, \psi_2, \psi_3, \psi_4)$. For the electromagnetic field we select as variables the components ϕ_μ of the four-potential. [19]

The Lagrangian is constructed additively out of three parts that correspond to the three designated fields (and simultaneously contain the field interactions).

Letting

$$E_{\mu\nu} = \frac{\partial \phi_\nu}{\partial x^\mu} - \frac{\partial \phi_\mu}{\partial x^\nu} \tag{88}$$

represent the electromagnetic field tensor, then the radiation term in the Lagrangian is [39]

$$\mathcal{E} = \frac{1}{4} E_{\mu\nu} \mathcal{E}^{\mu\nu}, \tag{89}$$

where we define

$$\mathcal{E}^{\mu\nu} = E^{\mu\nu} h',$$

where h' is the determinant of $h_{i,\nu}$ and the $E^{\mu\nu}$ are the contravariant components of the tensor $E_{\mu\nu}$.

In order to construct the matter term we fix a special system of Pauli matrices [20]

$$\rho_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \rho_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \rho_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{90}$$

Thus let us set

$$\begin{cases} \alpha_{\bar{l}} = -i \begin{pmatrix} \rho_{\bar{l}} & 0 \\ 0 & -\rho_{\bar{l}} \end{pmatrix} \ (\bar{l} = 1,2,3), \\ \qquad \alpha_4 = 1. \end{cases} \tag{91}$$

[16] V. Fock, Zeit. f. Phys. **57**, p. 261, (1930)
[17] H. Weyl, Zeit. f. Phys. **56**, p. 330, (1929)
[18] Berliner Berichte 1929, p. 137.
[19] Since we have set $x^4 = ct$ we have $\phi_4 = -\phi$, where ϕ represents the scalar potential. [38]
[20] They differ from the Fock matrices only slightly. The essentially different feature of the specialization is $\sigma_4 = 1$ (In Fock's notation $\sigma_0 = 1$).

We now introduce the notation

(92) $$e_{\overline{k}} = -1, \;\; e_4 = 1,$$

so the matrices α_i are Hermitian and they have the commutation property

(93) $$\alpha_m \alpha_k e_k + \alpha_k \alpha_m e_m = 2 e_m \delta_{mk}.$$

We still need the matrix

(94) $$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(where the ones represent two-row identity matrices). [40]

With regard to the latin indices a sum over doubly repeated indices is understood, whereby the factor e_k is to be ignored when counting the number of indices. In addition to the $h_{i,\nu}$ the contravariant $h_i{}^\nu$ also appear, and they satisfy the relations [41]

(95) $$\begin{cases} h_k^\nu h_{l,\nu} = e_k \delta_{kl}, \\ e_k h_k^\nu h_{k,\mu} = \delta_\mu^\nu, \end{cases}$$

which express the orthogonality of the tetrads in the space with metric [42]

(96) $$g_{\mu\nu} = e_k h_{k,\mu} h_{k,\nu}.$$

Defining

(97) $$\eta_{\rho\sigma}^l = \frac{\partial h_{l,\rho}}{\partial x^\sigma} - \frac{\partial h_{l,\sigma}}{\partial x^\rho},$$

and

(98) $$2\gamma_{mkl} = \underline{(\eta_{\rho\sigma}^l h_m^\sigma h_k^\rho + \eta_{\rho\sigma}^m h_l^\sigma h_k^\rho + \eta_{\rho\sigma}^k h_m^\sigma h_l^\rho) h'}, \text{ [43]}$$

(99) $$C_l = \frac{1}{4} e_k \alpha_m \alpha_k \gamma_{mkl} + \frac{e}{\omega} \phi_\sigma h_l^\sigma h', \;\; (\omega = \frac{hc}{2\pi i}), \text{ [44]}$$

and finally [45]

(100) $$\gamma^\sigma = e_k \alpha_k h_k^\sigma h',$$

then the matter term in the Lagrangian reads [46]

(101) $$\Re \omega \psi^* \left(\gamma^\sigma \frac{\partial \psi}{\partial x^\sigma} - e_l \alpha_l C_l \psi \right) - mc^2 \psi^* \sigma \psi h'.$$

(x^* is the complex conjugate of x. $\Re x$ is the real part of x. $\Im x$ is the imaginary part of x.)

Now we have (cf. Fock, *loc. cit.*, formula (24)),

$$e_l(\alpha_l C_l + C_l^\dagger \alpha_l) = -\frac{\partial \gamma^\sigma}{\partial x^\sigma},$$

and consequently

$$\Im \omega \psi^* \left(\gamma^\sigma \frac{\partial \psi}{\partial x^\sigma} - e_l \alpha_l C_l \psi\right) = \frac{\omega}{2} \frac{\partial}{\partial x^\sigma}(\psi^* \gamma^\sigma \psi). \tag{102}$$

We can therefore take as the matter term [47]

$$\mathcal{W} = \omega \psi^* \left(\gamma^\sigma \frac{\partial \psi}{\partial x^\sigma} - e_l \alpha_l C_l \psi\right) - mc^2 \psi^* \sigma \psi h', \tag{103}$$

instead of (101).

For the gravitational part we take $\frac{1}{2\chi}\mathcal{G}$, where $\chi = \frac{8\pi f}{c^2}$ (f = Newton's gravitational constant) and [48]

(104)

$$\mathcal{G} = e_k e_l \eta^l_{\rho\sigma} h^\rho_l h^{\rho'}_k g^{\sigma\sigma'} h' \eta^k_{\rho'\sigma'} - \frac{1}{2} e_k e_l \eta^l_{\rho\sigma} h^{\rho'}_l h^\rho_k g^{\sigma\sigma'} h' \eta^k_{\rho'\sigma'} - \frac{1}{4} e_l \eta^l_{\rho\sigma} g^{\sigma\sigma'} g^{\rho\rho'} h' \eta^l_{\rho'\sigma'};$$

as is easily checked (cf. eg. Weyl, *loc. cit.*) that $\mathcal{G}$ differs from scalar curvature density $\mathcal{R}$ by a divergence

$$\mathcal{R} = \mathcal{G} - 2\frac{d}{dx^\nu}\underline{\left(e_l h^\nu_l \frac{\partial(h^\sigma_l h')}{\partial x^\sigma}\right)}. \tag{105}$$

Altogether we have therefore

$$\mathcal{L} = \frac{1}{2\chi}\mathcal{G} + \mathcal{E} + \mathcal{W}. \tag{106}$$

In contrast to the usual form of relativity theory where the field quantities $g_{\mu\nu}$ were not vectors but were rather second rank tensors, the two constituent parts $\mathcal{G}$ and $2\frac{d}{dx^\nu}\underline{\left(e_l h^\nu_l \frac{\partial(h^\sigma_l h')}{\partial x^\sigma}\right)}$ of $\mathcal{R}$ in (105) are scalar densities under the general relativistic transformation group. On the other hand $\mathcal{G}$ is not by itself gauge invariant, but $\mathcal{R}$ is. [49]

§12. The gauge invariance group

The simplest group admitted by our function $\mathcal{L}$ is the gauge invariance group for which the x^ν and the $h_{l,\nu}$ remain invariant while the ϕ_ν and ψ transform as follows [50]

$$\left\{ \begin{array}{l} \delta\phi_\nu = \frac{\partial \xi}{\partial x^\nu}, \\ \delta\psi = -\frac{e}{\omega}\xi\psi. \end{array} \right. \tag{107}$$

Under this group $\delta\mathcal{L} = 0$.

In order to ease the comparison with the general theory we set [21]

$$\phi_\nu = Q_\nu,\ \psi = Q_5,$$

so that we have

$$c^\mu_\nu = \delta^\mu_\nu,\ c^\mu_5 = 0$$

with the single resulting identity

$$\mathcal{P}^4 = 0. \tag{108}$$

This follows directly from the explicit calculation of the $\mathcal{P}^{\alpha\nu}$:

$$\begin{cases} \mathcal{P}^{\mu\nu} = \mathcal{E}^{\nu\mu}, \\ \mathcal{P}^{5\nu} = \omega\psi^*\gamma^\nu. \end{cases} \tag{109}$$

In order to discuss this simple example further we first disregard gravitation, i.e., we set $h_{i,\nu} = \delta_{i,\nu}$. [76]

The Hamiltonian then takes the form

$$\mathcal{H} = \mathcal{H}_0 + \lambda\mathcal{P}^4, \tag{110}$$

where $\mathcal{H}_0$ is for example the special Hamiltonian function selected in H. P. II that does not contain $\mathcal{P}^4$. [52]

The field equations read [53]

$$\begin{cases} \omega\dot{Q}_{\overline{\nu}} = [\overline{\mathcal{H}}_0, Q_{\overline{\nu}}], \\ \dot{Q}_4 = \lambda, \\ \omega\dot{Q}_5 = [\overline{\mathcal{H}}_0, Q_5]; \end{cases} \tag{111}$$

$$\omega\dot{\mathcal{P}}^\alpha = [\overline{\mathcal{H}}_0, \mathcal{P}^\alpha],\ (\alpha = 1, \cdots, 5); \tag{112}$$

Since we also have $j = 1$ we have a constraint [22] besides (108) [54]

$$[\overline{\mathcal{H}}_0, \mathcal{P}^4] = 0. \tag{113}$$

So λ in (111) remains fundamentally undetermined and the fourth equation (112) is replaced by (113).

The infinitesimal transformation $\overline{\mathcal{M}}$ reads here:

$$\overline{\mathcal{M}} = \int \left\{ \frac{\partial\xi}{\partial x^\nu}\mathcal{E}^{4\nu} - e\xi\psi^*\gamma^4\psi \right\} dx^1 dx^2 dx^3,$$

or through integration by parts

$$\overline{\mathcal{M}} = \int \left\{ \frac{\partial\xi}{\partial x^4}\mathcal{P}^4 - \xi\left[\frac{\partial\mathcal{E}^{4\overline{\nu}}}{\partial x^{\overline{\nu}}} + e\psi^*\gamma^4\psi \right] \right\} dx^1 dx^2 dx^3,$$

[21] Since the ψ are not Hermitian it is necessary to make some small modifications in order to adapt these variables to the formalism. The is no need to go into these details here.

[22] In the notation of H. P. II (113) reads $C = 0$.

The square bracket is nothing other than $\frac{1}{\omega}[\overline{\mathcal{H}}_0, \mathcal{P}^4]$, or $\dot{\mathcal{P}}^4$, so that

$$\overline{\mathcal{M}} = \int \left\{ \frac{\partial \xi}{\partial x^4} \mathcal{P}^4 - \xi \frac{d\mathcal{P}^4}{dx^4} \right\} dx^1 dx^2 dx^3, \tag{114}$$

in agreement with (63′).

According to the general theory $\mathcal{P}^4 = 0$ must hold due to the field equations and the identities; this is in fact the continuity equation of electricity. [55]

§13. General relativistic covariance

For an arbitrary coordinate transformation [56]

$$\delta x^\nu = \xi^\nu, \tag{115}$$

$$\delta h_{l,\nu} = -h_{l,\mu} \frac{\partial \xi^\mu}{\partial x^\nu}, \tag{115′}$$

and

$$\left\{ \begin{array}{c} \delta \phi_\nu = -\phi_\mu \frac{\partial \xi^\mu}{\partial x^\nu}, \\ \delta \psi = 0. \end{array} \right. \tag{115″}$$

The Lagrangian behaves as a scalar density under this transformation. [57]

We retain the notation of the previous paragraphs for the momenta conjugate to ϕ_ν and ψ, and we represent the momenta $\mathcal{P}^{\alpha\nu}$ conjugate to the $h_{l,\nu}$ by $\mathcal{P}_l^{\nu\mu}$ So the inproper identities (28) read in the present case

$$\underline{\phi_\rho(\mathcal{E}^{\mu\nu} + \mathcal{E}^{\nu\mu}) + h_{i,\rho}(\mathcal{P}_i^{\nu\mu} + \mathcal{P}_i^{\mu\nu})} = 0;$$

taking into account that $\mathcal{E}^{\mu\nu} + \mathcal{E}^{\nu\mu} = 0$ they reduce to [58]

$$\mathcal{P}_i^{\nu\mu} + \mathcal{P}_i^{\mu\nu} = 0, \tag{116}$$

and we thus follow the four proper identities

$$\mathcal{P}_l^4 = 0. \tag{117}$$

The direct calculation in fact gives (116) since $\mathcal{G}$ and $\mathcal{R}$ depend on the $h_{i,\nu,\mu}$ only through the $\eta^i_{\tau\sigma}$ and

$$\frac{\partial \eta^i_{\rho\sigma}}{\partial h_{i,\nu,\mu}} = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma,$$

i.e., is antisymmetric in μ and ν. One finds [59] [60]

$$\left\{ \begin{array}{l} \mathcal{P}_i^{\nu\mu} = \underline{\eta^i_{\rho\sigma} g^{\nu\sigma} g^{\mu\rho} + e_l \eta^l_{\rho\sigma}(g^{\sigma\mu} h_i^\nu - g^{\sigma\nu} h_i^\mu)} \\ \qquad -\underline{e_l \eta^l_{\rho\sigma} h_i^\rho (h_l^\nu g^{\sigma\mu} - h_l^\mu g^{\sigma\nu}) e_i h'} \frac{1}{2\chi} \\ \qquad -\Re \frac{\omega}{4} \psi^* \alpha_l \alpha_m \alpha_k \psi e_l e_k \frac{\partial \gamma_{mkl}}{\partial h_{i,\nu,\mu}}, \end{array} \right. \tag{118}$$

The infinitesimal transformation $\overline{\mathcal{M}}$ takes the form [61]

$$\overline{\mathcal{M}} = -\int dx^1 dx^2 dx^3 \left\{ \frac{\partial \xi^\mu}{\partial x^4}(h_{i,\mu}\mathcal{P}_i^4 + \phi_\mu \mathcal{P}^4) - \xi^\mu \left[\frac{\partial}{\partial x^{\overline{\nu}}} \underline{(h_{i,\mu}\mathcal{P}_i^{\overline{\nu}} + \phi_\mu \mathcal{P}^{\overline{\nu}})} + \mathcal{G}_\mu \right] \right\};$$

le t us consider in particular the translation $\xi^\mu = \epsilon^\mu = \text{const}$ so that $\frac{d\overline{\mathcal{M}}}{dx^4} = 0$ yields the energy-momentum conservation law

$$\overline{\mathcal{G}}_\mu = \text{const};$$

For a linear transformation we derive from $\overline{\mathcal{M}}$ the generalization of the angular momentum conservation law (cf. H. P. II, p. 177).

According to the general theory (section 7) the setting of the coefficients of ξ^μ equal to zero yields a constraint

$$\mathcal{G}_\mu + \frac{\partial}{\partial x^{\overline{\nu}}} \underline{(h_{i,\mu}\mathcal{P}_i^{\overline{\nu}} + \phi_\mu \mathcal{P}^{\overline{\nu}})} = 0. \tag{119}$$

F. Klein [23] has already shown in another context that (119) is equivalent to the four field equations.

§14. The true Beincovariance

For this group we have [62]

$$\delta x^\nu = 0, \quad \delta \phi_\mu = 0,$$

$$\delta h_{i,\nu} = e_k \xi_{ik} h_{k,\nu}, \quad (\xi_{ik} = -\xi_{ki}), \tag{120}$$

and based on (120) it is easy to show that [63]

$$\delta \psi = \frac{1}{4} e_k \xi_{ik} \alpha_i \alpha_k \psi. \tag{120'}$$

We have here an example of the "second case" treated in section 9, since $\mathcal{G}$ is only locally invariant and only $\mathcal{R}$ is truly Beininvariant; $\mathcal{M}$ and $\mathcal{E}$ are also true Beininvariants]].

For the purpose of calculating formula (74), $\mathcal{I}_r^\mu \equiv \mathcal{I}_{(ik)}^\mu$, it will be useful to temporarily select as a variable

$$Q_l^\alpha \equiv h' h_l^\alpha.$$

Then according to (105) [64]

$$f_l^{\nu,\alpha\rho} = -\frac{1}{\chi} e_l h_l^\nu \delta_\rho^\alpha,$$

[23] Gött. Nach. 1918, p. 185

and according to (120) [65]

$$c_{l,\alpha(ik)} = \delta_{il} e_k h_k^\alpha h' - \delta_{kl} e_i h_i^\alpha h'.$$

Referring now to (71) one easily finds that [66]

$$\mathcal{I}^\nu_{(ik)} = \frac{1}{\chi} e_i e_k \frac{d}{dx^\rho} (h' h_i^\nu h_k^\rho - h' h_i^\rho h_k^\nu). \tag{121}$$

To calculate $\mathcal{F}_r \equiv \mathcal{F}_{(ik)}$ it will be convenient to return to the original variables $Q_{l,\alpha} \equiv h_{l,\alpha}$ and $Q_5 = \psi$. Then we have [67]

$$c_{l,\alpha(ik)} = \delta_{il} e_k h_{k,\alpha} - \delta_{kl} e_i h_{i,\alpha},$$

and in addition, by (120′), we set [68]

$$c_{5(ik)} = \frac{1}{4} (e_k \alpha_i \alpha_k \psi - e_i \alpha_k \alpha_i \psi).$$

We have accordingly [69]

$$\mathcal{F}_{(ik)} = \mathcal{P}_i^\nu e_k h_{k,\nu} - \mathcal{P}_k^\nu e_i h_{i,\nu} + \frac{\omega}{4} e_k e_l h' h_l^4 \psi^* (\alpha_l \alpha_i \alpha_k - \alpha_l \alpha_k \alpha_i) \psi.$$

Now according to (98)

$$\frac{\partial \gamma_{mjl}}{\partial h_{i,\nu,4}} e_k h_{k,\nu} - \frac{\partial \gamma_{mjl}}{\partial h_{k,\nu,4}} e_i h_{i,\nu} = h' h_l^4 (\delta_{im} \delta_{kj} - \delta_{ij} \delta_{km});$$

then according to (118) we set

$$\mathcal{P}_i^\nu = \tilde{\mathcal{P}}_i^\nu - \Re \frac{\omega}{4} e_l \psi^* \alpha_l \alpha_m \alpha_j \psi e_j \frac{\partial \gamma_{mjl}}{h_{i,\nu,4}}, \tag{122}$$

where $\tilde{\mathcal{P}}_i^\nu$ represents the momentum in the absence of matter. Then the $\mathcal{F}_{(ik)}$ simplifies, as it must, to

$$\mathcal{F}_{(ik)} = \underline{\tilde{\mathcal{P}}_i^\nu e_k h_{k,\nu} - \tilde{\mathcal{P}}_k^\nu e_i h_{i,\nu}}. \tag{123}$$

According to (121) and (123) the six proper identities read

$$\underline{\tilde{\mathcal{P}}_i^\nu e_k h_{k,\nu} - \tilde{\mathcal{P}}_k^\nu e_i h_{i,\nu}} + \frac{1}{\chi} e_i e_k \frac{d}{dx^\rho} (h' h_i^4 h_k^\rho - h' h_i^\rho h_k^4) = 0, \tag{124}$$

which can also be obtained directly from (118).

§15. Supplementary observations on the gravitational and matter fields

1. After having sketched in the previous paragraphs how the Fock-Weyl one-body theory can be quantized, we would like to briefly address a point that is

treated differently by Fock and by Weyl, namely the construction of the energy-momentum tensor $\mathcal{T}_i{}^\nu$ of matter. The Fock approach leads to a non-symmetric tensor and seems for our purposes to be inappropriate. We prefer the Weyl definition [70]

$$\mathcal{T}_i{}^\nu = \frac{\delta\mathcal{W}}{\delta h_{i,\nu}} \tag{125c}$$

that based on the equations of motion gives a symmetric tensor. However, since Fock works with a two-component ψ while we want to stay with Weyl's four-component theory, it would not be redundant to repeat here *mutatis mutandis* the Weyl calculation of $\mathcal{T}_i{}^\nu$.

The symmetry of $\mathcal{T}_i{}^\nu$ follows immediately from $\delta\mathcal{W} = 0$, where δ is the variation (120), (120'), for it follows from setting to zero the coefficients of $\xi_{(ik)}$ that [71]

$$\mathcal{T}_i{}^\nu e_k h_{k,\nu} - \mathcal{T}_k{}^\nu e_i h_{i,\nu} = -\frac{1}{2}\Re\frac{\delta\mathcal{W}}{\delta\psi}(e_k\alpha_i\alpha_k - e_i\alpha_k\alpha_i)\psi,$$

i.e., [72]

$$\mathcal{T}_i{}^\nu e_k h_{k,\nu} - \mathcal{T}_k{}^\nu e_i h_{i,\nu} = 0$$

using the field equations

$$\frac{\delta\mathcal{W}}{\delta\psi} = 0 \quad \text{and} \quad \frac{\delta\mathcal{W}}{\delta\psi^*} = 0.$$

This equation expresses the fact that the tensor [73]

$$\mathcal{T}''_{ik} = e_i e_k \mathcal{T}_i^\nu h_{k,\nu}$$

is symmetric under the interchange of i and k.

Instead of (125c) we can just as well set

$$\mathcal{T}_i{}^\nu = \frac{\delta\Re\mathcal{W}}{\delta h_{i,\nu}}, \tag{126c}$$

which gives us a real tensor $\mathcal{T}_i{}^\nu$. It is more convenient to calculate [74]

$$\mathcal{T}'_{i,\nu} = \frac{\delta\Re\mathcal{W}}{\delta h_i^\nu} = -e_k h_{k,\nu} h_{k,\rho}\mathcal{T}_k^\rho \equiv e_i h' T_{i,\nu} \tag{127c}$$

Based on (103) we find [75]

$$\left\{\begin{array}{c} \mathcal{T}'_{i,\nu} = \Re\omega\psi^*\alpha_i\frac{\partial\psi}{\partial x^\nu} - e\psi^*\alpha_i\psi\phi_\nu - h_{i,\nu}W \\ +\Re\frac{\omega}{4}e_k h_k^\rho h_{m,\nu}\frac{\partial}{\partial x^\rho}\{\psi^*\alpha_i\alpha_m\alpha_k\psi\} \\ -\Re\frac{\omega}{4}e_i e_l e_k\psi^*\alpha_l\alpha_m\alpha_k\psi\left\{\frac{\partial\gamma_{mkl}}{\partial h_i^\nu} - \frac{\partial}{\partial x^\rho}\frac{\partial\gamma_{mkl}}{\partial h_{i,\rho}^\nu}\right\}, \\ \text{with } W = \frac{1}{h'}\mathcal{W}. \end{array}\right. \tag{128c}$$

We confine ourselves now to special relativity in setting [76]

$$h_i^\nu = e_i h_{i,\nu} = \delta_{i\nu}.$$

Then (128c) becomes [24] [77]

$$\text{(129c)} \quad \mathcal{T}'_{i,\nu} = \Re\omega\psi^*\alpha_i\frac{\partial\psi}{\partial x^\nu} - \delta_{i\nu}W - e\psi^*\alpha_i\psi\phi_\nu + \Re\frac{\omega}{4}e_\rho e_\nu\frac{\partial}{\partial x^\rho}(\psi^*\alpha_i\alpha_\nu\alpha_\rho\psi),$$

with

$$W = \Re\omega e_\rho\psi^*\alpha_\rho\frac{\partial\psi}{\partial x^\rho} - mc^2\psi^*\sigma\psi - e_\rho e\psi^*\alpha_\rho\phi_\rho.$$

In particular we then have [78]

$$\mathcal{T}'_{44} = \Re\omega\psi^*\alpha_{\overline{\rho}}\frac{\partial\psi}{\partial x^{\overline{\rho}}} - e\psi^*\alpha_{\overline{\rho}}\psi\phi_{\overline{\rho}} + mc^2\psi^*\sigma\psi,$$

i.e., the energy operator is [79]

$$\text{(130c)} \qquad H = \alpha_{\overline{\rho}}\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^{\overline{\rho}}} - \frac{e}{c}\phi_{\overline{\rho}}\right) + mc\sigma,$$

In addition we have [80]

$$\mathcal{T}'_{4\overline{\nu}} = \Re\omega\psi^*\frac{\partial\psi}{\partial x^{\overline{\nu}}} - c\psi^*\psi\phi_{\overline{\nu}} + \Re\frac{\omega}{4}\frac{\partial}{\partial x^{\overline{\rho}}}(\psi^*\alpha_{\overline{\nu}}\alpha_{\overline{\rho}}\psi);$$

we set

$$\text{(131)} \qquad \alpha_1\alpha_2 = \mu_3,$$

for cyclical permutations of the indices $\{ijk\}$, then we have for example

$$\mathcal{T}'_{41} = \Re\omega\psi^*\frac{\partial\psi}{\partial x^1} - e\psi^*\psi\phi_1 + \frac{\omega}{4}\left\{\frac{\partial}{\partial x^2}(\psi^*\mu_3\psi) - \frac{\partial}{\partial x^3}(\psi^*\mu_2\psi)\right\}.$$

The momentum operator is therefore:

$$\text{(132c)} \qquad P_{\overline{\nu}} = \frac{h}{2\pi i}\frac{\partial}{\partial x^{\overline{\nu}}} - \frac{e}{c}\phi_{\overline{\nu}};$$

furthemore we get for the angular momentum:

$$M_1 = x^2T'_{43} - x^3T'_{42} = \Re\omega\psi^*\left(x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2}\right)\psi - c\psi^*\psi[x^2\phi_3 - x^3\phi_2]$$

$$+\frac{\omega}{4}\left\{x^2\frac{\partial}{\partial x^1}(\psi^*\mu_2\psi) - x^2\frac{\partial}{\partial x^2}(\psi^*\mu_1\psi) - x^3\frac{\partial}{\partial x^3}(\psi^*\mu_1\psi) + x^3\frac{\partial}{\partial x^1}(\psi^*\mu_3\psi)\right\},$$

[24] cf. also H. Tetrode, Zeit. f. Phys. **60**, p. 858, 1928. Fomulae (13) and (16) as well as the text on p. 862.

and consequently for the corresponding operator

$$M_1 = \frac{h}{2\pi i}\left(x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2}\right) - \frac{e}{c}(x^2\phi_3 - x^3\phi_2) + \frac{i\mu_1}{2}. \tag{133c}$$

2. In the preceeding we have assumed Bose-Einstein statistics for the tetrads $h_{i,\nu}$, i.e., we have chosen a minus sign in the bracket symbol for the CR's. One could ask whether it would be possible to apply Fermi statistics to the tetrads. The criterion for the admissibility of CR's with a plus sign is the following (cf. H.P. I, p. 29): the *usual* bracket symbol (with the minus sign) $[\mathcal{G}_\mu, Q_\alpha]$, $[\mathcal{G}_\mu, \mathcal{P}^\alpha]$ must assume the same value when one replaces the minus sign in $[Q_\alpha, Q_\beta]$, $[\mathcal{P}^\alpha, \mathcal{P}^\beta]$, and $[Q_\alpha, \mathcal{P}^\beta]$ with a plus sign.

Applying this criterion the answer with reference to the tetrads is *no*, since one sees from the form of the Hamiltonian (quadratic in the $\mathcal{P}^\alpha$) that in the transition from a plus to a minus sign $[\overline{\mathcal{H}}_0, Q_\alpha]$ undergoes a change; the two terms quadratic in the the $\mathcal{P}^\alpha$ in the bracket symbol are different, and the changes do not compensate each other.

3. The pure (vacuum) gravitational field could be described by the $g_{\mu\nu}$ instead of the $h_{i,\nu}$. Then we would be dealing with another variation of the "second case" and due to the general covariance group we would obtain four identities of the form $(\mathcal{P}^\alpha + \mathcal{R}^{\alpha 4})c^4_{\alpha\nu} = 0$.

Summary

1. When the Lagrangian function $\mathcal{L}(Q_\alpha; \dot{Q}_\alpha)$ transforms under the group [25]

$$\left\{ \begin{aligned} \delta x^\nu &= \alpha_r^{\nu,0}(x)\xi^r(x), \\ \delta Q_\alpha &= c^0_{\alpha r}(x, Q)\xi^r + c^\sigma_{\alpha r}\tfrac{\partial \xi^r}{\partial x^\sigma}, \end{aligned} \right. \tag{2'}$$

as a scalar density, then there arises between the Q and the conjugate momenta $\mathcal{P}$ the identities

$$\mathcal{F}_r \equiv \underline{\mathcal{P}^\alpha c^4_{\alpha r}} = 0. \tag{29'}$$

In case $\mathcal{L} + \mathcal{L}'$, but not $\mathcal{L}$, is a scalar density, where $\mathcal{L}'$ is linear in the second derivatives of the Q_α, then $\mathcal{P}^\alpha + \mathcal{R}^{\alpha 4}$ appears everywhere in place of $\mathcal{P}^\alpha$.

2. Consequently the solution of the equations

$$\mathcal{P}^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{Q}_\alpha}$$

for the $\dot{Q}_\alpha$ takes the form

$$\dot{Q}_\alpha = \dot{Q}^0_\alpha(\mathcal{P}, Q) + \lambda^r c^4_{\alpha r}, \tag{31'}$$

[25] For the purpose of this overview we specialize the formulas to the physically interesting case $j = 1$.

with arbitrary spacetime functions λ^r.

The Hamiltonian thereby takes the form

$$\mathcal{H} = \mathcal{H}_0(\mathcal{P}, Q) + \lambda^r \mathcal{F}_r. \tag{35'}$$

The basic equations of the theory are the canonical field equations, the canonical CR's, and the constraints

$$\mathcal{F}_r = 0 \text{ and } \frac{d\mathcal{F}_r}{dx^4} = 0.$$

3. The infinitesimal transformations of the group can be expressed as

$$\omega\delta^*\Phi = [\overline{\mathcal{M}}, \Phi]. \tag{45}$$

$$\mathcal{M} = \underline{\mathcal{P}^\alpha \delta Q_\alpha} - \mathcal{G}_\mu \delta^\mu. \tag{46}$$

(Φ is an arbitrary function that depends only on Q and $\mathcal{P}$; $\mathcal{G}_\mu$ is the (pseudo) energy-momentum density).

A special case of $\overline{\mathcal{M}}$ on an arbitrary slice $x^4 = x^4_0$ is $\overline{\epsilon^r \mathcal{F}_r}$. It follows from $\mathcal{F}_r = 0$ that the $\mathcal{F}_r$ commute amongst themselves, i.e., that the constraints $\mathcal{F}_r = 0$ are compatible.

Furthermore, due to the field equations,

$$\frac{d\overline{\mathcal{M}}}{dx^4} = 0, \tag{58}$$

from which it follows that

$$\overline{\mathcal{M}} = \int dx^1 dx^2 dx^3 \left\{ \mathcal{F}_r \frac{\partial \xi^r}{\partial x^4} - \frac{d\mathcal{F}_r}{dx^4} \xi^r \right\}, \tag{63'}$$

and

$$\frac{d^2 \mathcal{F}_r}{(dx^4)^2} \equiv 0, \tag{64'}$$

based on the field equations (temporal evolution of the constraints).

4. The basic system of equations is invariant under the group.

5. The electromagnetic field, the Dirac material field, and the gravitational field including all interactions were treated as examples. The relevant groups are the gauge invariance group, the true Bein covariance group, and the group of general relativity theory.

In particular, as regards gravitation, it is not possible to quantize the corresponding field quantities with Fermi statistics.

I express my sincere thanks to Prof. Pauli for his suggestion to undertake this work and for his valuable advice.

Zurich, Physics Institute of the Swiss Federal Institute of Technology, March 5, 1930

(received March 18, 1930)

References

[1] An ellipsis is missing in these equations. they should read

$$\begin{cases} \delta x^\nu = a_r^{\nu,0}(x)\xi^r(x) + a_r^{\nu,\sigma}(x)\frac{\partial\xi^r}{\partial x^\sigma} + \ldots + a_r^{\nu,\sigma\cdots\tau}(x)\frac{\partial^k\xi^r}{\partial x^\sigma\cdots\partial x^\tau}, \\ \delta Q_\alpha = c^0_{\alpha r}(x,Q)\xi^r(x) + c^\sigma_{\alpha r}(x,Q)\frac{\partial\xi^r}{\partial x^\sigma} + \ldots + c^{\sigma\cdots\tau}_{\alpha r}(x,Q)\frac{\partial^j\xi^r}{\partial x^\sigma\cdots\partial x^\tau}. \end{cases}$$

[2] Rosenfeld defines the variation of Q_α as follows, letting a "prime" represent the transformed variable,

$$\delta Q_\alpha := Q'_\alpha(x+\delta x) - Q_\alpha(x).$$

Below he will call this a "local" variation.

[3] Note that with these definitions Rosenfeld is accomodating the inclusion of transformations for which $\delta x^\nu = 0$, in which case he is evidently taking $k=-1$. Thus the local Lorentz transformations treated in Part 2 will also satisfy the condition

$$j \geq k+1,$$

where $k=-1$ and $j=0$. See equation (120).

[4] The δ^* variations are minus the Lie derivative in the direction δx^ν. E. Nöther denotes these variations in the functional form by $\bar{\delta}$ in Nachr. v. d. Ges. d. Wiss. zu Göttingen, 235 (1918). P. G. Bergmann, beginning in 1949, Phys. Rev. **75**, 680, continues Nöther's use of the $\bar{\delta}$ notation. These variations are now called "active" variations.

[5]

$$\delta^*\left(\frac{d\Phi}{dx^\nu}\right) := \Phi'_{,\nu}(x'-\delta x) - \Phi(x) = \delta(\Phi_{,\nu}) - \Phi_{,\nu\mu}\delta x^\mu.$$

Also

$$(\delta^*\Phi)_{,\nu} := (\Phi(x'-\delta x) - \Phi(x))_{,\nu} = (\delta\Phi)_{,\nu} - \Phi_{,\nu\mu}\delta x^\mu - \Phi_{,\mu}\delta x^\mu_{,\nu}.$$

Setting these two expressions equal we find

$$\delta(\Phi_{,\nu}) = (\delta\Phi)_{,\nu} - \Phi_{,\mu}\delta x^\mu_{,\nu}.$$

[6] A scalar density R of weight one is defined to transform under the coordinate transformation $x'^\nu(x)$ as

$$\mathcal{R}'(x') = \mathcal{R}(x)\det\left(\frac{\partial x}{\partial x'}\right),$$

So for $x'^\nu = x^\nu + \delta x^\nu(x)$,

$$\mathcal{R}'(x') = \mathcal{R}(x)\left(1 - \frac{\partial\delta x^\nu}{\partial x^\nu}\right)$$

resulting in equation (8)

[7] Anderson and Bergmann called these 'primary' identities (or primary constraints when referring to phase space) in Phys. Rev. **83**, 1018 (1951) Theirs is now the conventional terminology.

[8] The commutator will not depend on either $\dot{Q}_\alpha$ or $\mathcal{P}^\alpha$.

[9] The matrix $\mathcal{A}^{\alpha\beta}$ formed from the second partial derivatives with respect to the variable velocities is known as the Legendre matrix. Rosenfeld has demonstrated that the $c_{\alpha r}^{4\cdots 4}$ constitute r_0 independent null vectors of this matrix. The existence of null vectors is now taken as a defining property of singular Lagrangian systems.

[10] The reference is to the rewritten form of equation (21) that follows equation (29).

[11] The line containing the commutators may be written as the nested commutator

$$\frac{1}{2}\left[\mathcal{A}^{\alpha\beta'}, [\mathcal{P}^{\gamma'} - \mathcal{D}^{\gamma'}, \mathcal{A}_{\gamma'\beta'}]\right]$$

This expression vanishes by virtue of the second implication.

[12] The construction of the c-number special solution of (21), and also the q-number solution with appropriate modifications, proceeds as follows: $\mathcal{A}^{\alpha\beta}$ is a real symmetric matrix, and hence may be diagonalized using the normalized eigenvectors $S^{\gamma\beta}$ satisfying

$$\mathcal{A}^{\alpha\beta} S^{\gamma\beta} = \lambda_\gamma S^{\gamma\alpha}, \text{ no sum over } \gamma,$$

where the eigenvalues λ_γ are real. The $S^{(N-r_0+r)\alpha} \equiv \frac{c_{r\alpha}}{\left(\sum c_{r\beta}^2\right)^{1/2}}$ are by virtue of (25) normalized eigenvectors with eigenvalue zero. Viewed as a transformation matrix the inverse of S is S^t. Then

$$S\mathcal{A}S^t = \begin{pmatrix} \bar{A} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\bar{A}$ is the $N - r_0$ dimensional diagonal matrix

$$\bar{A} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & \lambda_{N-r_0} \end{pmatrix}$$

Let us represent quantities transformed under S with a "bar", so for example we have

$$\bar{\mathcal{A}} = S\mathcal{A}S^t = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \vdots & \vdots & 0 & 0 & \vdots & 0 \\ 0 & 0 & \cdots & \lambda_{N-r_0} & 0 & \cdots & 0 \\ 0 & \cdots & & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & & & & & 0 \end{pmatrix}.$$

We let $1 \leq \bar{\alpha}' \leq N - r_0$ and $N - r_0 < \bar{\alpha}'' \leq N$, and it follows from (25) and (26) that

$$\mathcal{P}^{\bar{\alpha}''} = \mathcal{D}^{\bar{\alpha}''} = 0,$$

i.e.,

$$\bar{\mathcal{P}} = S\mathcal{P} = \begin{pmatrix} \bar{\mathcal{P}}^1 \\ \vdots \\ \bar{\mathcal{P}}^{N-r_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{\mathcal{P}}' \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and

$$\bar{\mathcal{D}} = S\mathcal{D} = \begin{pmatrix} \bar{\mathcal{D}}^1 \\ \vdots \\ \bar{\mathcal{D}}^{N-r_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{\mathcal{D}}' \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore according to (21) we have the trivial solutions

$$\dot{\bar{Q}}^0_{\alpha''} = 0.$$

Next, using the inverse

$$\bar{A}^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ 0 & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & \frac{1}{\lambda_{N-r_0}} \end{pmatrix},$$

we find

$$\dot{\bar{Q}}^0_{\alpha'} = \left(\bar{A}^{-1}\right)^{\alpha'\beta'} \left(\bar{\mathcal{P}}^{\beta'} - \bar{\mathcal{D}}^{\beta'}\right),$$

so the complete particular solution is

$$\dot{\bar{Q}}^0 = S\dot{Q}^0 = \begin{pmatrix} \bar{A}^{-1}\left(\bar{P}' - \bar{D}'\right) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\bar{Q}}'^0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

Thus we have

$$\dot{Q}^0_\alpha = (\mathcal{A}^{-1})^{\alpha\beta}\left(P^\beta - D^\beta\right),$$

where I have defined

$$(\mathcal{A}^{-1})^{\alpha\beta} := S^{\delta'\alpha}(\bar{A}^{-1})^{\delta'\gamma'}S^{\gamma'\beta}.$$

The general solution for $\dot{Q}_\alpha$ is

$$\dot{Q}_\alpha = \dot{Q}^0_\alpha + \lambda^r c_{r\alpha}.$$

[13] This classical equality is established by Rosenfeld in Ann. de l'I. H. P. (1932), 25 as follows: With $\mathcal{L} = \mathcal{L}(Q, Q_{,\bar{\nu}}, \dot{Q}(Q, Q_{,\bar{\mu}}, \mathcal{P}))$,

$$\left.\frac{\partial \mathcal{L}}{\partial Q_{\alpha,\bar{\nu}}}\right|_{\mathcal{P}} = \left.\frac{\partial \mathcal{L}}{\partial Q_{\alpha,\bar{\nu}}}\right|_{\dot{Q}} + \frac{\partial \mathcal{L}}{\partial \dot{Q}_\beta}\left.\frac{\partial \dot{Q}_\beta}{\partial Q_{\alpha,\bar{\nu}}}\right|_{\mathcal{P}},$$

or

$$\left.\frac{\partial \mathcal{L}}{\partial Q_{\alpha,\bar{\nu}}}\right|_{\dot{Q}} = \left.\frac{\partial \mathcal{L}}{\partial Q_{\alpha,\bar{\nu}}}\right|_{\mathcal{P}} - \frac{\partial \mathcal{L}}{\partial \dot{Q}_\beta}\left.\frac{\partial \dot{Q}_\beta}{\partial Q_{\alpha,\bar{\nu}}}\right|_{\mathcal{P}}.$$

On the other hand

$$\frac{\partial \mathcal{H}}{\partial Q_{\alpha,\bar{\nu}}} = \mathcal{P}^\beta\left.\frac{\partial \dot{Q}_\beta}{\partial Q_{\alpha,\bar{\nu}}}\right|_{\mathcal{P}} - \left.\frac{\partial \mathcal{L}}{\partial Q_{\alpha,\bar{\nu}}}\right|_{\mathcal{P}} = -\left.\frac{\partial \mathcal{L}}{\partial Q_{\alpha,\bar{\nu}}}\right|_{\dot{Q}}.$$

[14] Using the notation and results from [12], we will find the explicit expression for the Hamiltonian $\mathcal{H} = \mathcal{P}^t\left(\dot{Q}^0 + \lambda^r c_r\right) - \mathcal{L}(Q, \dot{Q}^0 + \lambda^r c_r)$. First we rewrite the Lagrangian density as

$$\mathcal{L}(Q, \dot{Q}^0 + \lambda^r c_r) = \frac{1}{2}\dot{Q}^t\mathcal{A}\dot{Q} + \mathcal{D}^t\dot{Q} + \mathcal{E},$$

where I define $\mathcal{E} := \mathcal{B}^{\alpha\bar{\nu}}Q_{\alpha,\bar{\nu}} + \mathcal{C}$. Using the fact that the c_r are null vectors of $\mathcal{A}$ and are orthogonal to $\mathcal{D}$, we have

$$\mathcal{L}(Q, \dot{Q}^0 + \lambda^r c_r) = \mathcal{L}(Q, \dot{Q}^0) = \frac{1}{2}\dot{\bar{Q}}^t S\mathcal{A}S^t\dot{\bar{Q}} + \bar{\mathcal{D}}^t\dot{\bar{Q}} + \mathcal{E}$$

$$= \frac{1}{2}\dot{\bar{Q}}_{\alpha'}A^{\alpha'\beta'}\dot{\bar{Q}}_{\beta'} + \bar{\mathcal{D}}^{\alpha'}\dot{\bar{Q}}_{\alpha'} + \mathcal{E}$$

$$= \frac{1}{2}\left(\bar{A}^{-1}\right)^{\alpha'\rho'}\left(\bar{\mathcal{P}}^{\rho'} - \bar{\mathcal{D}}^{\rho'}\right) A^{\alpha'\beta'}\left(\bar{A}^{-1}\right)^{\beta'\sigma'}\left(\bar{\mathcal{P}}^{\sigma'} - \bar{\mathcal{D}}^{\sigma'}\right) + \bar{\mathcal{D}}^{\alpha'}\left(\bar{A}^{-1}\right)^{\alpha'\rho'}\left(\bar{\mathcal{P}}^{\rho'} - \bar{\mathcal{D}}^{\rho'}\right) + \mathcal{E}$$

$$= \frac{1}{2}\bar{\mathcal{P}}^{\alpha'}\left(\bar{A}^{-1}\right)^{\alpha'\beta'}\bar{\mathcal{P}}^{\beta'} - \frac{1}{2}\bar{\mathcal{D}}^{\alpha'}\left(\bar{A}^{-1}\right)^{\alpha'\beta'}\bar{\mathcal{D}}^{\beta'} + \mathcal{E}$$

$$= \frac{1}{2}\mathcal{P}^t\mathcal{A}^{-1}\mathcal{P} - \frac{1}{2}\mathcal{D}^t\mathcal{A}^{-1}\mathcal{D} + \mathcal{E}.$$

Therefore

$$\mathcal{H} = \mathcal{H}_0 + \lambda^r\mathcal{F}_r,$$

where

$$\mathcal{H}_0 = \frac{1}{2}\mathcal{P}^t\mathcal{A}^{-1}\mathcal{P} - \mathcal{D}^t\mathcal{A}^{-1}\mathcal{P} + \frac{1}{2}\mathcal{D}^t\mathcal{A}^{-1}\mathcal{D} - \mathcal{E}$$

[15] Referring to [14]

$$\dot{Q}_\alpha = \frac{\partial H}{\partial P^\alpha} = \left(P^\gamma - D^\gamma\right)(\mathcal{A}^{-1})^{\gamma\alpha} + \lambda^r c_{r\alpha} = \dot{Q}^0_\alpha + \lambda^r c_{r\alpha}$$

Generally the preservation of the primary constraint under time evolution will lead to a fixation of the functions λ^r and/or lead to more constraints. Rosenfeld addresses this question below.

Also, referring to [12], note that

$$\lambda^r\mathcal{F}_r = \lambda^r c_{r\alpha}\mathcal{P}^\alpha = \left(\sum c^2_{r\beta}\right)^{1/2}\lambda^r e^\alpha_{N-r_0+r}\mathcal{P}^\alpha$$

$$= \left(\sum c^2_{r\beta}\right)^{1/2}\lambda^r S^{\alpha\, N-r_0+r}\mathcal{P}^\alpha = \left(\sum c^2_{r\beta}\right)^{1/2}\lambda^r\bar{\mathcal{P}}^{N-r_0+r}.$$

In other words, the additional term in the Hamiltonian is equal to a linear sum of the vanishing momentum linear combinations $\bar{\mathcal{P}}^{\alpha''}$

[16] If the spatial boundary is taken to be finite it appears to be sufficient for Rosenfeld to assume that the field quantities take the same constant value at each coordinate boundary. See my remark [17] preceding equation (44). On the other hand if Rosenfeld is contemplating a falloff behavior at spatial infinity he needs to assume that the Lagrangian asymptotically approaches zero. An alternative not mentioned by Rosenfeld would be to treat a spatially compact manifold.

[17]

$$\left[\left[\overline{\mathcal{G}}_\mu, \overline{\mathcal{G}}_\nu\right], \Phi\right] = \omega^2\left(\frac{d}{dx^\mu}\frac{d\Phi}{dx^\nu} - \frac{d}{dx^\nu}\frac{d\Phi}{dx^\mu}\right) = 0.$$

[18] There is a crucial error in the following discussion of the generator of infinitesimal transformations that will render invalid some of the properties that Rosenfeld derives. I will present some details later with regard to the example that is treated in Part Two. The results that Rosenfeld obtains from this point on are strictly valid only for Yang-Mills type local gauge theories. They do not hold for generally covariant systems.

The fundamental problem is that we are asked to conceive of the quantum mechanical commutator $[\overline{\mathcal{M}}, \Phi]$ as being placed in correspondence with a classical Poisson bracket, to be understood therefore as a function of canonical phase space variables. Thus we must interpret equations (47) as representing on the left hand side the variations of the canonical variables, initially conceived as functions of configuration variables Q_α and velocities $Q_{\alpha,4}$, and then projected under the Legendre transformation to functions of Q_α, $Q_{\alpha,\bar{\nu}}$, and $\mathcal{P}^\alpha$. (This is especially clear in the derivation of the variation of $\mathcal{P}^\alpha$ in equation (50); $\mathcal{P}^\alpha$ is understood here to be a function of Q_α and $Q_{\alpha,\mu}$).

The problem is that some configuration-velocity space functions $f(Q_\alpha, Q_{\alpha,\bar{\nu}}, Q_{\alpha,0}$ are mapped under the Legendre transformation to zero. The functions with this property are namely those for which

$$c_{\alpha,r}\frac{\partial}{\partial \dot{Q}_\alpha} f = 0$$

To prove this assertion, define

$$f(Q_\alpha, Q_{\alpha,\bar{\nu}}, Q_{\alpha,0}) := \tilde{f}(Q, \mathcal{P}(Q, \dot{Q})).$$

Then since $c_{\alpha,r}\frac{\mathcal{P}_\beta}{\partial \dot{Q}_\alpha} = 0$ we find

$$c_{\alpha,r}\frac{\partial}{\partial \dot{Q}_\alpha} f = c_{\alpha,r}\frac{\partial \tilde{f}}{\partial \mathcal{P}^\beta}\frac{\mathcal{P}_\beta}{\partial \dot{Q}_\alpha} = 0 \tag{134}$$

Thus only projectable functions can appear on the left hand side of equations (47). Rosenfeld does not recognize this necessary restriction.

This restriction was first mentioned in print, as far as I can determine, by Bergmann and Brunings, "Non-Linear Field Theories II. Canonical Equations and Quantization", Rev. Mod. Phys., 480 (1949). They give the condition (134) explicitly in their equation (3.24). The projectability obstacle was rediscovered by Wald and Lee in 1990, "Local Symmetries and Constraints", J. Math. Phys., 725 (1990). It is possible to overcome this obstacle by introducing a coordinate symmetry transformation group whose elements depend on the metric. Pons, Salisbury, and Shepley showed in 1996 that the compulsory metric dependence on infinitesimal coordinate transformations is precisely the decomposition of transformations into those that are tangential to a given time foliation of the spacetime manifold (that do not necessarily depend on the metric), and transformations that are perpendicular to the foliation (and therefore do depend on the metric).

Returning to Rosenfeld, to be consistent he needed to restrict his variations $\delta^* Q_\alpha$ to those that satisfy

$$c_{\beta,r}\frac{\partial}{\partial \dot{Q}_\beta}\delta^* Q_\alpha = 0.$$

[19] We have

$$\frac{1}{\omega}\left[\overline{\mathcal{M}},\mathcal{P}^\alpha\right]=\frac{1}{\omega}\left[\int d^3x'(\mathcal{P}'^\beta\delta Q'_\beta-\mathcal{G}'_\mu\delta x'^\mu),\mathcal{P}^\alpha\right]$$

$$=\underline{-\mathcal{P}^\beta\frac{\partial\delta Q_\beta}{\partial Q_\alpha}}-\frac{1}{\omega}\left[\int d^3x'\left(\mathcal{P}'^\beta Q'_{\beta,\bar{\mu}}\delta x'^{\bar{\mu}}+\mathcal{H}'\delta x'^4\right),\mathcal{P}^\alpha\right]$$

$$=\underline{-\mathcal{P}^\beta\frac{\partial\delta Q_\beta}{\partial Q_\alpha}}-\frac{\partial}{\partial x^{\bar{\mu}}}\left(\mathcal{P}^\alpha\delta x^{\bar{\mu}}\right)+\frac{\partial\mathcal{H}}{\partial Q_\alpha}\delta x^4-\frac{\partial}{\partial x^{\bar{\mu}}}\left(\delta x^4\frac{\partial\mathcal{H}}{\partial Q_{\alpha,\bar{\mu}}}\right)$$

But according to (33) (and the fact that $\frac{\partial\mathcal{H}}{\partial Q_\alpha}=-\frac{\partial\mathcal{L}}{\partial Q_\alpha}$) we have

$$\frac{\partial\mathcal{H}}{\partial Q_\alpha}-\frac{\partial}{\partial x^{\bar{\mu}}}\frac{\partial\mathcal{H}}{\partial Q_{\alpha,\bar{\mu}}}=-\frac{\partial\mathcal{L}}{\partial Q_\alpha}+\frac{\partial}{\partial x^{\bar{\mu}}}\frac{\partial\mathcal{L}}{\partial Q_{\alpha,\bar{\mu}}}=-\dot{\mathcal{P}}^\alpha,$$

and therefore

$$\frac{1}{\omega}\left[\overline{\mathcal{M}},\mathcal{P}^\alpha\right]=\underline{-\mathcal{P}^\beta\frac{\partial\delta Q_\beta}{\partial Q_\alpha}}-\frac{\partial}{\partial x^{\bar{\mu}}}\left(\mathcal{P}^\alpha\delta x^{\bar{\mu}}\right)+\frac{\partial\delta x^4}{\partial x^{\bar{\mu}}}\mathcal{P}^{\alpha\bar{\nu}}-\dot{\mathcal{P}}^\alpha\delta x^4.$$

[20]

$$\delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial Q_\beta}\delta Q_\beta+\frac{\partial\mathcal{L}}{\partial Q_{\beta,\mu}}\delta(Q_{\beta,\mu}),$$

so

$$\frac{\partial\delta\mathcal{L}}{\partial Q_{\alpha,\nu}}=\frac{\partial^2\mathcal{L}}{\partial Q_\beta\partial Q_{\alpha,\nu}}\delta Q_\beta+\frac{\partial^2\mathcal{L}}{\partial Q_{\beta,\mu}\partial Q_{\alpha,\nu}}\delta(Q_{\beta,\mu})+\frac{\partial\mathcal{L}}{\partial Q_{\beta,\mu}}\frac{\partial\delta(Q_{\beta,\mu})}{\partial Q_{\alpha,\nu}},$$

where we used the fact that δQ_β does not depend on $Q_{\alpha,\nu}$. We also have

$$\delta\left(\frac{\partial\mathcal{L}}{\partial Q_{\alpha,\nu}}\right)=\frac{\partial^2\mathcal{L}}{\partial Q_{\alpha,\nu}\partial Q_\beta}\delta Q_\beta+\frac{\partial^2\mathcal{L}}{\partial Q_{\alpha,\nu}\partial Q_{\beta,\mu}}\delta(Q_{\beta,\mu}).$$

Thus we find

$$\delta\mathcal{P}^{\alpha\nu}=\frac{\partial^2\mathcal{L}}{\partial Q_{\alpha,\nu}\partial Q_\beta}\delta Q_\beta-\mathcal{P}^{\beta\mu}\frac{\partial\delta(Q_{\beta,\mu})}{\partial Q_{\alpha,\nu}}.$$

[21] Since $\delta x^\mu=0$ and $\delta Q_\alpha=c^{4\ldots4}_{\alpha r}\frac{\partial^j\xi^r}{(\partial x^4)^j}$

$$\overline{\mathcal{M}}=\int d^3x\mathcal{P}^\alpha\delta Q_\alpha=\overline{\epsilon^r\mathcal{F}_r}$$

[22] More generally,

$$[\mathcal{F}_s(\vec{x},x^4_0),\mathcal{F}_r(\vec{x}',x^4_0)]=\delta^3(\vec{x}-\vec{x}')c^t_{rs}\mathcal{F}_t(\vec{x},x^4_0).$$

Recall that Rosenfeld is confining himself here to transformations for which $\delta x^\mu=0$.

[23]

$$\frac{d\mathcal{M}^\nu}{dx^\nu} = \frac{d\mathcal{P}^{\alpha\nu}}{dx^\nu}\delta Q_\alpha + \mathcal{P}^{\alpha\nu}\frac{d\delta Q_\alpha}{dx^\nu} - \frac{d\mathcal{P}^{\alpha\nu}}{dx^\nu}Q_{\alpha,\mu}\delta x^\mu - \mathcal{P}^{\alpha\nu}Q_{\alpha,\mu\nu}\delta x^\mu$$
$$-\mathcal{P}^{\alpha\nu}Q_{\alpha,\mu}\frac{d\delta x^\mu}{dx^\nu} + \frac{\partial\mathcal{L}}{\partial Q_\alpha}Q_{\alpha,\mu}\delta x^\mu + \mathcal{P}^{\alpha\nu}Q_{\alpha,\mu\nu}\delta x^\mu + \mathcal{L}\frac{d\delta x^\mu}{dx^\mu}$$

Recall that

$$\delta Q_{\alpha,\nu} = \frac{d\delta Q_\alpha}{dx^\nu} - Q_{\alpha,\mu}\frac{d\delta x^\mu}{dx^\nu}.$$

and according to (13)

$$\mathcal{L}\frac{d\delta x^\mu}{dx^\mu} = -\delta\mathcal{L} = -\frac{\partial\mathcal{L}}{\partial Q_\alpha}\delta Q_\alpha - \mathcal{P}^{\alpha\nu}\delta Q_{\alpha,\nu}.$$

Substituting we find

$$\frac{d\mathcal{M}^\nu}{dx^\nu} = -\frac{\partial\mathcal{L}}{\partial Q_\alpha}\delta Q_\alpha + \frac{\partial\mathcal{L}}{\partial Q_\alpha}Q_{\alpha,\mu}\delta x^\mu + \frac{d\mathcal{P}^{\alpha\nu}}{dx^\nu}\delta Q_\alpha - \frac{d\mathcal{P}^{\alpha\nu}}{dx^\nu}Q_{\alpha,\mu}\delta x^\mu = -\mathcal{L}^\alpha\delta^* Q_\alpha$$

[24] Rosenfeld evidently assumes that the variations vanish on the spatial boundary.

[25] According to (38) and (43), replacing Φ in (43) by $\delta^*\Phi$, we have

$$\omega^2\frac{d\delta^*\Phi}{dx^\mu} = \left[\overline{\mathcal{G}_\mu}, [\overline{\mathcal{M}}, \Phi]\right] + \omega\left[\frac{\partial\overline{\mathcal{M}}}{\partial x^\mu}, \Phi\right],$$

where in the last term we recognized that since Φ contains no explicit x dependence,

$$\omega\frac{\partial}{\partial x^\mu}\delta^*\Phi = \left[\frac{\partial\overline{\mathcal{M}}}{\partial x^\mu}, \Phi\right].$$

But according to (5),

$$\omega^2\frac{d\delta^*\Phi}{dx^\mu} = \omega^2\delta^*\left(\frac{d\Phi}{dx^\mu}\right) = \left[\overline{\mathcal{M}}, [\overline{\mathcal{G}_\mu}, \Phi]\right],$$

and therefore, using the Jacobi identity,

$$\left[\overline{\mathcal{M}}, [\overline{\mathcal{G}_\mu}, \Phi]\right] - \left[\overline{\mathcal{G}_\mu}, [\overline{\mathcal{M}}, \Phi]\right] = \left[[\overline{\mathcal{M}}, \overline{\mathcal{G}_\mu}], \Phi\right] = \omega\left[\frac{\partial\overline{\mathcal{M}}}{\partial x^\mu}, \Phi\right].$$

We conclude that

$$\left[\frac{d\overline{\mathcal{M}}}{dx^\nu}, \Phi\right] = 0.$$

[26] There is a puzzle here. As Rosenfeld notes, the requirement that arbitrary time derivatives of the constraints $\mathcal{F}_r = 0$ vanish is an internal consistency requirement. It is independent of the analysis of the generator $\mathcal{M}$ that he has just undertaken. With this requirement in mind, the result that $\frac{d^{j+1}\mathcal{F}_r}{(dx^4)^{j+1}} = 0$ may be viewed as a consistency check. What Rosenfeld has actually proven here is that on account of the requirement that $\frac{d^i\mathcal{F}_r}{(dx^4)^i} = 0$, the generator $\mathcal{M}$ must vanish. But he never says this explicitly.

[27] For a simple example suppose that $\mathcal{F}_r$ has no explicit time dependence. Then in requiring that $\frac{d\mathcal{F}_r}{(dx^4)} = 0$ we find

$$[\overline{\mathcal{H}_0}, \mathcal{F}_r] = \chi_r + b_r^s \mathcal{F}_s,$$

where χ_r is a secondary constraint (if it does not vanish identically). Then

$$\omega^2 \frac{d^2\mathcal{F}_r}{(dx^4)^2} = [\overline{\mathcal{H}}, \chi_r + (b_r^s + c_{rt}^s \mathcal{F}_s] = [\overline{\mathcal{H}}_0, \chi_r] + \lambda^u [\mathcal{F}_u, \chi_r],$$

where I have omitted terms that are proportional to $\mathcal{F}$.

[28] We are to understand the presence of arbitrary functions as Rosenfeld's definition of "missing".

[29] We are still considering here the special case $j = 1$.

[30] Keep in mind that in this definition $k+1 \leq i \leq j$, so these derivatives need not vanish.

[31] Rosenfeld is demonstrating here that the cumulative variation in Φ obtained by first performing an infinitesimal transformation generated by $\overline{\mathcal{N}}$ and then followed by a transformation generated by $\overline{\mathcal{M}}$ can be written as a transformation generated by $\overline{\mathcal{M}}$ followed by a transformation generated by an altered generator, namely $\overline{\mathcal{N}} + \frac{1}{\omega}[\overline{\mathcal{M}}, \overline{\mathcal{N}}]$ The transformed fields under the first transformation are labeled by a "prime", whereas the transformed fields under the second transformation are denoted with a "tilde". Thus

$$\Phi' = \Phi + \frac{1}{\omega}[\overline{\mathcal{N}}, \Phi],$$

and to be unambiguous Rosenfeld should denote the second transformed field as $\widetilde{(\Phi')}$ and not $(\tilde{\Phi}')$, Thus

$$\widetilde{(\Phi')} = \Phi' + \frac{1}{\omega}[\overline{\mathcal{M}}, \Phi'] = \Phi + \frac{1}{\omega}[\overline{\mathcal{N}}, \Phi] + \left[\overline{\mathcal{M}}, \Phi + \frac{1}{\omega}[\overline{\mathcal{N}}, \Phi]\right]$$

$$= \Phi + \frac{1}{\omega}[\overline{\mathcal{N}}, \Phi] + \frac{1}{\omega}[\overline{\mathcal{M}}, \Phi] - \frac{1}{\omega^2}\left([\Phi, [\overline{\mathcal{N}}, \overline{\mathcal{M}}]] + [\overline{\mathcal{N}}, [\Phi, \overline{\mathcal{M}}]]\right)$$

$$= \Phi + \frac{1}{\omega}[\overline{\mathcal{M}}, \Phi] + \frac{1}{\omega}\left[\overline{\mathcal{N}} + \frac{1}{\omega}[\overline{\mathcal{M}}, \overline{\mathcal{N}}], \Phi + \frac{1}{\omega}[\overline{\mathcal{M}}, \Phi]\right]$$

$$= \tilde{\Phi} + \frac{1}{\omega}\left[\overline{\mathcal{N}} + \frac{1}{\omega}[\overline{\mathcal{M}}, \overline{\mathcal{N}}], \tilde{\Phi}\right],$$

where in the second line the Jacobi identity was used and in the third line we ignore terms of order $\left(\frac{1}{\omega}\right)^3$. Equivalently, he has computed the generator of the commutator of the two transformations generated by $\overline{\mathcal{N}}$ and $\overline{\mathcal{M}}$; this infinitesimal commutator is generated by $\frac{1}{\omega}[\overline{\mathcal{M}}, \overline{\mathcal{N}}]$.

[32] In other words, the commutator must be equal to some (so far unknown) linear combination of the generators of the invariant subgroup.

[33] Rosenfeld does not employ the identity (81) to prove (80′). The key observation here is that the coefficients of $Q_{\alpha,\nu\rho}$ in $r^{\rho}_{r,\rho}$ must vanish, where $r^{\rho}_{r} := r^{\alpha\rho} c_{\alpha r}$. But since

$$r^{\rho}_{r,\rho} = \frac{\partial r^{\rho}_{r}}{\partial Q_{\alpha}} + \frac{\partial r^{\rho}_{r}}{\partial Q_{\alpha,\nu}} Q_{\alpha,\nu\rho},$$

the desired coefficient is

$$\frac{\partial r^{\rho}_{r}}{\partial Q_{\alpha,\nu}} + \frac{\partial r^{\nu}_{r}}{\partial Q_{\alpha,\rho}} = 0.$$

This is (80′).

[34]
$$\frac{\partial}{\partial Q_{\beta,\rho}} \left(\frac{d f^{\nu,\alpha\mu}}{dx^{\mu}} \right) = \frac{\partial f^{\nu,\alpha\rho}}{\partial Q_{\beta}} - \frac{d}{dx^{\mu}} \left(\frac{\partial f^{\nu,\alpha\mu}}{\partial Q_{\beta,\rho}} \right) = \frac{\partial f^{\nu,\alpha\rho}}{\partial Q_{\beta}}$$

[35] Recall (80).

[36] Because of (80) $[\mathcal{R}^{\alpha 4}, Q_{\alpha}] = 0$.

[37] We will begin here in translating Rosenfeld's notation into conventional contemporary form. He is using Fock's conventions regarding the Vierbeine, and these were in turn adopted from Levi-Civita, *Sitzsungsberichte der Preussischen Akademie der Wissenschaft*, 137 (1929). He employs a Minkowski metric with signature $(+1 -1 -1 -1)$. We will denote Minkowski indices with capitalized latin letters from the middle of the alphabet, so the components of the Minkowski metric are

$$\eta_{IJ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then $h_{i,\nu}$ is the covariant tetrad with the Minkowski index lowered: $h_{i,\nu} = e_{I\mu}$, where i becomes a Minkowski index ranging from 0 to 3. Thus e_k essentially raises Minkowski indices. Lower-case Greek indices will continue to represent coordinate indices.

[38] Note that with the chosen signature $\phi^0 = \phi_0 = -\phi$.

[39] Note that since $h_{i,\nu} = e_{I\mu}$, $\det(h_{i,\nu}) = -\det(e^{I}_{\nu}) = -(-g)^{1/2}$. Also note that Rosenfeld is employing Gaussian units, and since the action involves an integral over ct, all of the contributions to the Lagrangian density should be divided by c.

[40] Excepting for a cyclic permutation of the Cartesian coordinate axes, the ρ_i are i times the conventional Pauli matrices σ^i. In particular,

$$\rho_1 = i\sigma^3,$$

$$\rho_2 = i\sigma^1,$$

and

$$\rho_3 = i\sigma^2.$$

We therefore have

$$-\sigma\alpha_1 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \Gamma^3,$$

$$-\sigma\alpha_2 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} = \Gamma^1,$$

and

$$-\sigma\alpha_3 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \Gamma^2.$$

Letting

$$\Gamma^0 := \sigma,$$

the Γ^I are the 4×4 Dirac matrices. The sign of the spatial matrices differs from the modern chiral representation, yet the Γ^I still satisfy the correct anti-commutation relation

$$\{\Gamma^I, \Gamma^J\} = 2\eta^{IJ}.$$

In the following when translating Rosenfeld's spinorial expressions into modern notation I will ignore the different identification of spatial coordinate axes and write

$$e_i\sigma\alpha_i = \Gamma^I.$$

[41] I have corrected an obvious typographical error in the second of equations (95).

$h_i{}^\nu$ in modern notation is E^μ_I, where to avoid confusion when considering specific components I use a capital letter to represent contravariant coordinate objects. So $e_k h_{k,\mu} = e^K_\mu$ and $h^\nu{}_k e_k h_{k,\mu} = \delta^\nu_\mu$ is the statement that $E^\nu_K e^K_\mu = \delta^\nu_\mu$.

[42] $g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$.

[43] This is the densitized Ricci rotation tensor in orthonormal basis, as we will now show. The Ricci rotation coefficients are

$$\omega_{\mu IJ} = E^\alpha_I \nabla_\mu e_{J\alpha}, \tag{135}$$

where

$$\nabla_\mu e_{J\alpha} := \partial_\mu e_{J\alpha} - \Gamma^\beta_{\alpha\mu} e_{J\beta}.$$

Expanding the Cristoffel symbols in terms of the tetrads we find

(136) $$\omega_\mu{}^{IJ} = E^{\alpha I} e^J_{[\alpha,\mu]} - E^{\alpha J} e^I_{[\alpha,\mu]} + E^{\alpha I} E^{\beta J} e_{\mu L} e^L_{[\alpha,\beta]},$$

and therefore

$$\omega_L{}^{MK} := E^\mu_L \omega_\mu{}^{MK} = E^\mu_L E^{\alpha M} e^K_{[\alpha,\mu]} + E^\mu_L E^{\alpha K} e^M_{[\mu,\alpha]} + E^{\alpha M} E^{\beta K} e_{L[\alpha,\beta]}$$

Rewriting in Rosenfeld's notation we find, recognizing that his $\eta^l_{\rho\sigma}$ defined in (97) is our $2e_{L[\rho,\sigma]}$,

$$\begin{aligned} 2h'\omega_{LMK} &= 2(E^\mu_L E^\alpha_M e_{K[\alpha,\mu]} + E^\mu_L E^\alpha_K e_{M[\mu,\alpha]} + E^\alpha_M E^\beta_K e_{L[\alpha,\beta]})h' \\ &= -(\eta^l_{\rho\sigma} h^\sigma_m h^\rho_k + \eta^m_{\rho\sigma} h^\sigma_l h^\rho_k + \eta^k_{\rho\sigma} h^\sigma_m h^\rho_l)h' = -2\gamma_{mkl} \end{aligned}$$

[44] This object is essentially the combined local Lorentz and $U(1)$ connection for Dirac spinors. In our new notation

$$\begin{aligned} C_l &:= \frac{1}{4} e_k \alpha_m \alpha_k \gamma_{mkl} + \frac{e}{\omega} h^\sigma_l \phi_\sigma h' \\ &= \frac{1}{4}\Gamma^0\Gamma^N \gamma_{0NL} - \frac{1}{4}\Gamma^0\Gamma^i\Gamma^0\Gamma^N \gamma_{iNL} - \frac{ie}{\hbar c}\phi_\sigma E^\sigma_L (-g)^{1/2} \\ &= \frac{(-g)^{1/2}}{4}\left(\Gamma^0\Gamma^i \omega_{L0i} - \Gamma^0\Gamma^i\omega_{Li0} - \Gamma^0\Gamma^i\Gamma^0\Gamma^j \omega_{Lij}\right) - \frac{ie}{\hbar c}\phi_\sigma E^\sigma_L(-g)^{1/2} \\ &= \frac{(-g)^{1/2}}{4}\Gamma^I\Gamma^J\omega_{LIJ} - \frac{ie}{\hbar c}\phi_\sigma E^\sigma_L (-g)^{1/2} =: C_L \end{aligned}$$

where I use a latin lower case letter from the middle of the alphabet to denote a spatial Minkowski index, and I made use of the anticommutator for the Γ matrices, and the antisymmetry of the connection ω_{LMN} under the interchange of M and N.

[45]

$$\gamma^\sigma = -\Gamma^0\Gamma^K E^\sigma_K (-g)^{1/2}$$

To avoid confusion I am letting Γ^M represent the Minkowski Dirac gamma matrices.

[46] The mass and the charge terms appear with the wrong sign in this expression, as they do also in the final form of the matter Lagrangian, equation (103).

$$-e_l\alpha_l C_l = -\Gamma^0\Gamma^L C_L = -\Gamma^0\Gamma^L \frac{(-g)^{1/2}}{4}\Gamma^I\Gamma^J\omega_{LIJ} + \Gamma^0\Gamma^L\frac{ie}{\hbar c}\phi_\sigma E^\sigma_L(-g)^{1/2}$$

So we find upon substitution that

$$\Re\omega\psi^*\left(\gamma^\sigma\frac{\partial\psi}{\partial x^\sigma} - e_l\alpha_l C_l\psi\right) - mc^2\psi^*\sigma\psi h'$$

$$= \Re \left(i\hbar c(-g)^{1/2} \psi^\dagger \Gamma^0 E^\mu_I \Gamma^I \left(\frac{\partial}{\partial x^\mu} + \Omega_\mu - i\frac{e}{\hbar c}\phi_\mu \right) \psi \right) + mc^2 \psi^\dagger \Gamma^0 \psi (-g)^{1/2},$$

where

$$\Omega_\mu := \frac{1}{4}\Gamma^I \Gamma^J \omega_{\mu IJ},$$

is the spinor connection, consistent with the Cristoffel connection, that respects the scalar and vector nature of $\psi^\dagger \gamma^0 \psi$ and $\psi^\dagger \gamma^0 E^\mu_I \gamma^I \psi$, respectively. It was first constructed independently by H. Weyl, Zeitschrift fr Physik, **56**, 330 (1929) and V. Fock, Zeitschrift fï Physik, **57**, 261 (1929). Both authors were attempting a geometric unification of Dirac's electron theory with gravity. See the article by E. Scholz, "Local spinor structures in V. Fock's and H. Weyl's work on the Dirac equation", physics/0409158, for a discussion of the historical importance of this work both in the unification program and in the development of gauge theories in general. For the relevance to gauge theory see also the article by N. Straumann, "Gauge principle and QCD", physics/0509116.

[47] The final form of Rosenfeld's matter Lagrangian is

$$\mathcal{W} = \omega \psi^* \left(\gamma^\sigma \frac{\partial \psi}{\partial x^\sigma} - e_l \alpha_l C_l \psi \right) - mc^2 \psi^* \sigma \psi h'$$

$$= i\hbar c(-g)^{1/2} \overline{\psi} E^\mu_L \Gamma^L \left(\frac{\partial}{\partial x^\mu} + \Omega_\mu - i\frac{e}{\hbar c}\phi_\mu \right) \psi + mc^2 \overline{\psi} \psi (-g)^{1/2}$$

where $\overline{\psi} := \psi^\dagger \Gamma^0$.

[48] Rosenfeld's gravitational Lagrangian is obtained in the following way. The curvature in terms of the Ricci rotation coefficients is

$$ {}^4R^{IJ}_{\mu\nu} = E^\mu_I E^\nu_J \left(\partial_\mu \omega^{IJ}_\nu - \partial_\nu \omega^{IJ}_\mu + \omega_\mu{}^I{}_L \omega_\nu{}^{LJ} - \omega_\nu{}^I{}_L \omega_\mu{}^{LJ} \right) \tag{137}$$

Then the scalar curvature density is

$$\begin{aligned} {}^4\mathcal{R} &:= (-g)^{\frac{1}{2}}\, {}^4R \\ &= 2(-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \partial_\mu \omega^{IJ}_\nu + (-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \left(\omega_\mu{}^I{}_L \omega_\nu{}^{LJ} - \omega_\nu{}^I{}_L \omega_\mu{}^{LJ} \right) \\ &= \nabla_\mu \left(2(-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \omega^{IJ}_\nu \right) - \nabla_\mu \left(2(-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \right) \omega^{IJ}_\nu \\ &+ (-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \left(\omega_\mu{}^I{}_L \omega_\nu{}^{LJ} - \omega_\nu{}^I{}_L \omega_\mu{}^{LJ} \right), \end{aligned} \tag{138}$$

where in the third line we recognized that the covariant derivative of the vector density of weight one is just the ordinary derivative.

Using

$$\nabla_\mu E^\nu_J = -\omega_{\mu J}{}^L E^\nu_L, \tag{139}$$

and $\nabla_\mu g_{\alpha\beta} = 0$ we find

$$\begin{aligned}\nabla_\mu \left(2(-g)^{\frac{1}{2}} E^\mu_I E^\nu_J\right) \omega^{IJ}_\nu &= 2(-g)^{\frac{1}{2}} \left(\nabla_\mu E^\mu_I E^\nu_J + E^\mu_I \nabla_\mu E^\nu_J\right) \omega^{IJ}_\nu \\ &= 2(-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \left(\omega_\mu{}^I{}_L \omega_\nu{}^{LJ} - \omega_\nu{}^I{}_L \omega_\mu{}^{LJ}\right). \qquad (140)\end{aligned}$$

Therefore (138) may be rewritten as

(141)
$$^4\mathcal{R} = \nabla_\mu \left(2(-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \omega^{IJ}_\nu\right) - (-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \left(\omega_\mu{}^I{}_L \omega_\nu{}^{LJ} - \omega_\nu{}^I{}_L \omega_\mu{}^{LJ}\right),$$

or

(142)
$$^4\mathcal{R} - \nabla_\mu \left(2(-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \omega^{IJ}_\nu\right) = -(-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \left(\omega_\mu{}^I{}_L \omega_\nu{}^{LJ} - \omega_\nu{}^I{}_L \omega_\mu{}^{LJ}\right)$$

This is Rosenfeld's $-\mathcal{G}$ as we now show.

Defining

$$\eta_{I\alpha\beta} := 2e_{I[\alpha,\beta]}, \qquad (143)$$

and referring to (136) we have

$$\omega^{IJ}_\mu = \frac{1}{2} E^{\alpha I} \eta^J_{\alpha\mu} - \frac{1}{2} E^{\alpha J} \eta^I_{\alpha\mu} + \frac{1}{2} E^{\alpha I} E^{\beta J} e_{\mu L} \eta^L_{\alpha\beta} \qquad (144)$$

and

$$E^\mu_I \omega_\mu{}^I{}_L = -\frac{1}{2} E^\mu_I E^\alpha_L \eta^I{}_{\alpha\mu} + \frac{1}{2} g^{\alpha\mu} E^\beta_L e_{\mu M} \eta^M{}_{\alpha\beta} = E^\beta_L E^\alpha_M \eta^M{}_{\alpha\beta}. \qquad (145)$$

Therefore

(146)
$$E^\mu_I \omega_\mu{}^I{}_L E^\nu_J \omega_\nu{}^{LJ} = -E^\beta_L E^\alpha_M \eta^M{}_{\alpha\beta} E^{\sigma L} E^\rho_N \eta^N{}_{\rho\sigma} = -g^{\beta\sigma} E^\alpha_M \eta^M{}_{\alpha\beta} E^\rho_N \eta^N{}_{\rho\sigma}.$$

Similarly, using (144), we find

$$2E^\mu_I \omega_\mu{}^{LJ} = E^\mu_I E^{\alpha L} \eta^J{}_{\alpha\mu} - E^\mu_I E^{\alpha J} \eta^L{}_{\alpha\mu} + E^{\alpha L} E^{\beta J} \eta_{I\alpha\beta}, \qquad (147)$$

and therefore

$$\begin{aligned}E^\mu_I E^\nu_J \omega_\nu{}^I{}_L \omega_\mu{}^{LJ} &= \frac{1}{4}\left(E^\mu_I E^{\alpha L} \eta^J{}_{\alpha\mu} - E^\mu_I E^{\alpha J} \eta^L{}_{\alpha\mu} + E^{\alpha L} E^{\beta J} \eta_{I\alpha\beta}\right) \times \\ &\quad \left(E^\nu_J E^{\rho I} \eta_{L\rho\nu} - E^\nu_J E^\rho_L \eta^I{}_{\rho\nu} + E^{\rho I} E^\sigma_L \eta_{J\rho\sigma}\right) \\ &= -\frac{1}{2} \eta_{M\alpha\beta} \eta^N{}_{\rho\sigma} g^{\alpha\rho} E^\beta_N E^{\sigma M} - \frac{1}{4} \eta_{M\alpha\beta} \eta^M{}_{\rho\sigma} g^{\alpha\rho} g^{\beta\sigma} \qquad (148)\end{aligned}$$

Thus substituting from (146) and (148) we have

$$\begin{aligned}&(-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \left(\omega_\mu{}^I{}_L \omega_\nu{}^{LJ} - \omega_\nu{}^I{}_L \omega_\mu{}^{LJ}\right) \\ = \; & -(-g)^{\frac{1}{2}} \left(g^{\beta\sigma} E^\alpha_M \eta^M{}_{\alpha\beta} E^\rho_N \eta^N{}_{\rho\sigma} - \frac{1}{2} \eta_{M\alpha\beta} \eta^N{}_{\rho\sigma} g^{\alpha\rho} E^\beta_N E^{\sigma M} - \frac{1}{4} \eta_{M\alpha\beta} \eta^M{}_{\rho\sigma} g^{\alpha\rho} g^{\beta\sigma}\right) \\ = \; & \mathcal{G} \qquad (149)\end{aligned}$$

Next compare the divergence term to Rosenfeld's. According to (144) we have

$$E^{\mu}_{J}\omega^{IJ}_{\mu} = -E^{\mu}_{J}\nabla_{\mu}E^{\alpha I}e^{J}_{\alpha} = -\nabla_{\mu}E^{\mu I}. \tag{150}$$

Therefore the vector density appearing in the divergence of Rosenfeld in (105) is

$$\begin{aligned} -2E^{\mu I}\left((-g)^{\frac{1}{2}}E^{\nu}_{I}\right)_{,\nu} &= -2E^{\mu I}\nabla_{\nu}\left((-g)^{\frac{1}{2}}E^{\nu}_{I}\right) \\ &= -2(-g)^{\frac{1}{2}}E^{\mu I}\nabla_{\nu}E^{\nu}_{I} = -2(-g)^{\frac{1}{2}}E^{\mu}_{I}E^{\nu}_{J}\omega^{IJ}_{\nu}. \end{aligned} \tag{151}$$

This is indeed the vector density whose divergence appears in (142).

[49] See [57]

[50] Note that in cgs units, the dimension of the four-potential ϕ_μ is $mass^{1/2}length^{1/2}time^{-1}$, and the dimension of the descriptor ξ is therefore $mass^{1/2}length^{3/2}time^{-1}$. The factor $\frac{e}{\omega}\xi$ is therefore dimensionless.

[51] This specialization to the flat case should read

$$h_{i,\nu} = e_i\delta_{i,\nu},$$

as it is correctly expressed immediately following equation (128c).

[52] In modern notation the flat spacetime Lagrangian density is

$$\mathcal{L}_{flat} = i\hbar c\bar{\psi}\Gamma^{\mu}\psi_{,\mu} - e\phi_{\mu}\bar{\psi}\Gamma^{\mu}\psi - mc^2\bar{\psi}\psi - \frac{1}{4}E^{\mu\nu}E_{\mu\nu}.$$

We find

$$\dot{\phi}_a = \mathcal{P}^a + \phi_{0,a},$$

and

$$\mathcal{P}_\psi = i\hbar c\psi^*.$$

Substituting, we have

$$\mathcal{H}_{0\,flat} = \mathcal{P}^a\dot{\phi}_a + \mathcal{P}_\psi\dot{\psi} - \mathcal{L}_{0\,flat}$$

$$= \frac{1}{2}\mathcal{P}^a\mathcal{P}^a + \frac{1}{4}E^{ab}E_{ab} + \mathcal{P}^a\phi_{0,a} + \underline{e\phi_\mu\bar{\psi}\Gamma^\mu\psi + mc^2\bar{\psi}\psi - i\hbar c\bar{\psi}\Gamma^a\psi_{,a}}$$

$$= \frac{1}{2}\mathcal{P}^a\mathcal{P}^a + \frac{1}{4}E^{ab}E_{ab} + \mathcal{P}^a\phi_{0,a} + \underline{\frac{ie}{\hbar c}\phi_\mu\mathcal{P}_\psi\Gamma^0\Gamma^\mu\psi - \frac{imc}{\hbar}\mathcal{P}_\psi\Gamma^0\psi - \mathcal{P}_\psi\Gamma^0\Gamma^a\psi_{,a}}$$

[53]

$$\omega\dot{\phi}_a = \omega\left(\mathcal{P}^a + \phi_{0,a}\right),$$

$$\omega\dot{\psi} = \omega\left(-\Gamma^0\Gamma^a\psi_{,a} - \frac{ie}{\hbar c}\phi_\mu\Gamma^0\Gamma^\mu\psi - \frac{imc}{\hbar}\psi\right),$$

$$\omega\dot{\mathcal{P}}^a = \omega\left(E^{ab}_{,b} - e\bar{\psi}\Gamma^a\psi\right).$$

[54] We get the secondary constraint

$$\left[\overline{\mathcal{H}_{0\,flat}}, \mathcal{P}^0\right] = 0 = -i\hbar c\left(e\psi^*\psi - \mathcal{P}^a_{,a}\right).$$

[55] The charge density is $\rho = e\psi^*\psi$. Also, according to the equation of motion for $\mathcal{P}^a$,

$$\dot{\mathcal{P}}^a_{,a} = E^{ab}_{ab} - \left(e\bar{\psi}\Gamma^a\psi\right)_{,a} = -j^a_{,a},$$

where $j^a = e\bar{\psi}\Gamma^a\psi$ is the current density. Therefore the condition $\ddot{\mathcal{P}}^0 = 0$ reads $\frac{\partial\rho}{\partial t} + j^a_{,a} = 0$. This relation is satisfied as a consequence of the equation of motion for ψ.

[56] This is the site of Rosenfeld's crucial error. To be consistent it is necessary to restrict attention to variations that are projectable under the Legendre transformation, as observed in [18]. One set of projection conditions in this case is that all configuation-velocity functions must be annihilated by the operator $\frac{\partial}{\partial \dot{e}^I_0}$. In particular we require that

$$\frac{\partial \delta^* e^J_\mu}{\partial \dot{e}^I_0} = 0, \tag{152}$$

where

$$\delta^* e^J_\mu = -e^J_\nu \xi^\nu_{,\mu} - e^J_{\mu,\nu}\xi^\nu \tag{153}$$

The offending term, $-e^J_{0,0}\xi^0$ can be eliminated by introducing infinitesimal coordinate variations that depend explicitly on e^I_0. The full details will be presented elsewhere. The result is that permissible, i.e., projectable transformations are of the form

$$\delta^\mu = \delta^\mu_a \epsilon^a + n^\mu \epsilon^0, \tag{154}$$

where

$$n^\mu = -\frac{g^{\mu 0}}{\left(-g^{00}\right)^{1/2}}, \tag{155}$$

is the normal to the constant coordinate time surfaces. The functions ξ^μ are arbitrary functions of the spacetime coordinates, and in general also of the tetrad components e^I_a.

It turns out that when additional gauge symmetries are present, as in Rosenfeld's model, even this metric field dependence is not sufficient to render projectable diffeomorphism-induced field variations; only a combination of diffeomorphism-induced and gauge is projectable. It turns out that infinitesimal local Lorentz transformations must be added to the general coordinate transformations in order to obtain projectable transformations. This was first pointed out - though from a different perspective and a different context - by Salisbury and Sundermeyer, "The local symmetries of the Einstein-Yang-Mills Theory as phses space transformations", Phys. Rev. D27, 757 (1983). In my commentary here I will confine myself to the vacuum gravitational theory.

[57] Under the infinitesimal coordinate transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}$, coordinate covectors v_{μ} transform as

$$v'_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} v_{\nu}(x) = v_{\mu}(x) - \xi^{\nu}_{,\mu} v_{\nu}(x),$$

so

$$\delta v_{\mu} := v'_{\mu}(x') - v_{\mu}(x) = -\xi^{\nu}_{,\mu} v_{\nu}.$$

It is straightforward to show that under this coordinate transformation

$$\delta \left(\frac{\partial}{\partial x^{\mu}} \left(2(-g)^{\frac{1}{2}} E^{\mu}_{I} E^{\nu}_{J} \omega^{IJ}_{\nu} \right) \right) = -\xi^{\nu}_{,\nu} \left(\frac{\partial}{\partial x^{\mu}} \left(2(-g)^{\frac{1}{2}} E^{\mu}_{I} E^{\nu}_{J} \omega^{IJ}_{\nu} \right) \right).$$

In other words, the term that is subtracted from the Ricci scalar density to form the gravitational Langrangian density $\mathcal{G}$ is itself a scalar density. Thus, as Rosenfeld observes, $\mathcal{G}$ is a scalar density under general coordinate transformations.

[58] Recall that these identities arise from the vanishing of coefficients of the highest derivatives of ξ^{μ} that appear in the identity expressing the scalar density nature of the Lagrangian density. In this case we are dealing with second derivatives, and the irrelevance of the order in which they are undertaken results in the symmetrization under exchange of μ and ν.

[59] Representing in modern notation the momenta conjugate to e^{I}_{μ} by $\mathcal{P}^{\mu}_{I}$ we have

$$\mathcal{P}^{\mu I} = \frac{1}{2\chi} \frac{\partial \mathcal{G}}{\partial \dot{e}_{I\mu}} + \frac{\partial \mathcal{M}}{\partial \dot{e}_{I\mu}},$$

and

$$\frac{\partial \mathcal{G}}{\partial \dot{e}_{I\mu}} = (-g)^{1/2} \left(-2g^{\beta\sigma} E^{\alpha}_{M} E^{\rho L} + g^{\alpha\rho} E^{\beta L} E^{\sigma}_{M} + \frac{1}{2} g^{\alpha\rho} g^{\beta\sigma} \delta^{L}_{M} \right) \eta^{M}_{\alpha\beta} \frac{\partial}{\partial \dot{e}_{I\mu}} \eta_{L\rho\sigma}.$$

Substituting

$$\frac{\partial}{\partial \dot{e}_{I\mu}} \eta_{L\rho\sigma} = 2\delta^{I}_{L} \delta^{[\mu}_{\rho} \delta^{0]}_{\sigma},$$

we find

$$\frac{\partial \mathcal{G}}{\partial \dot{e}_{I\mu}} = (-g)^{1/2} \left(-4g^{\beta[0} E^{\mu]I} E^{\alpha}_{M} + 2g^{\alpha[\mu} E^{0]}_{M} E^{\beta I} + g^{\alpha[\mu} g^{0]\beta} \delta^{I}_{M} \right) \left(e^{M}_{\alpha,\beta} - e^{M}_{\beta,\alpha} \right).$$

In addition, the contribution to the momentum from the matter Lagrangian is

$$\frac{\partial \mathcal{M}}{\partial \dot{e}_{I\mu}} = i\hbar (-g)^{1/2} \bar{\psi} E^{\nu}_{L} \Gamma^{L} \frac{\partial \Omega_{\nu}}{\partial \dot{e}_{I\mu}} \psi.$$

Referring to (143),

$$\frac{\partial \Omega_\nu}{\partial \dot{e}_{I\mu}} = E^\alpha_K \delta^I_J \delta^\mu_{[\alpha}\delta^0_{\nu]} - E^\alpha_J \delta^I_K \delta^\mu_{[\alpha}\delta^0_{\nu]} + E^\alpha_K E^\beta_J e^L_\nu \delta^I_L \delta^\mu_{[\alpha}\delta^0_{\nu]}$$

$$= \delta^I_J E^{[\mu}_K \delta^{0]}_\nu - \delta^I_K E^{[\mu}_J \delta^{0]}_\nu + E^{[\mu}_K E^{0]}_J e^I_\nu,$$

and therefore

$$\frac{\partial \mathcal{M}}{\partial \dot{e}_{I\mu}} = 2i\hbar(-g)^{1/2}\bar{\psi}\Gamma^L \left(\Gamma^K\Gamma^I E^{[\mu}_K E^{0]}_L - E^{I[\mu}E^{0]}_L + \frac{1}{2}\Gamma^K\Gamma^J E^{[\mu}_K E^{0]}_J \delta^I_L\right)\psi.$$

The momentum conjugate to $e_{I\mu}$ is therefore given by

$$\mathcal{P}^{\mu I} = \frac{1}{\chi}(-g)^{1/2}\left(-2g^{\beta[0}E^{\mu]I}E^\alpha_M + g^{\alpha[\mu}E^{0]}_M E^{\beta I} + \frac{1}{2}g^{\alpha[\mu}g^{0]\beta}\delta^I_M\right)\left(e^M_{\alpha,\beta} - e^M_{\beta,\alpha}\right)$$

$$+2i\hbar(-g)^{1/2}\bar{\psi}\Gamma^L \left(\Gamma^K\Gamma^I E^{[\mu}_K E^{0]}_L - E^{I[\mu}E^{0]}_L + \frac{1}{2}\Gamma^K\Gamma^J E^{[\mu}_K E^{0]}_J \delta^I_L\right)\psi$$

[60] I doubt that Rosenfeld succeeded in deriving the canonical Hamiltonian $\mathcal{H}_0$; this is the problem of solving the momentum relations for the (non-gauge) velocities. He had made what turned out to be an important first step from the point of view of conventional Hamiltonian gravity. He had chosen to work with a Lagrangian that did not depend on the velocities $\dot{e}^I_0$, with the result that some of the corresponding primary constraints were trivial. Dirac first made this simplification in conventional gravity in 1958 when he succeeded in removing the time derivatives $\dot{g}^{0\mu}$ in " The theory of gravitation in Hamiltonian form", Proc. Roy. Soc. 246, 333 (1958). The determination of the velocities $\dot{g}_{ab}$ in terms of the remaining momenta was then greatly simplified, and he was able to easily write down the (vanishing) canonical Hamiltonian. Beginning in 1959 Arnowitt, Deser, and Misner made a similar discovery in a new Palatini approach to gravity, summarized in "The dynamics of general relativity", in Gravitation: an introduction to current research, L. Witten, ed. (Wiley, New York, 1962). This paper is available online at arXiv:gr-qc/0405109. I will derive Rosenfeld's $\mathcal{H}_0$ elsewhere.

[61] This expression is incorrect, as is Rosenfeld's derivation of secondary constraints that follows. I will discuss the corrections that must be made in another publication. As mentioned above, the problems stem from the failure to take Legendre projectability into account.

[62] $\delta e_{I\nu} = \xi_{IJ} e^J_\nu$

[63] Recall that since $\alpha_{\bar{i}} = -\Gamma^0\Gamma^{\bar{i}}$, $\alpha_4 = 1$, $\sigma = \Gamma^0$, and $e_{\bar{i}}\sigma\alpha_{\bar{i}} = \Gamma^{\bar{i}}$,

$$\delta\psi = \frac{1}{4}e_j\xi_{ij}\alpha_i\alpha_j\psi = \frac{1}{4}\xi_{\bar{i}\bar{j}}\Gamma^0\Gamma^{\bar{i}}\Gamma^0\Gamma^{\bar{j}}\psi - \frac{1}{2}\xi_{0\bar{j}}\Gamma^0\Gamma^{\bar{j}}\psi = -\frac{1}{4}\xi_{IJ}\Gamma^I\Gamma^J\psi$$

[64] Recall that according to [48]

$$\frac{1}{2\chi}\mathcal{G} = \frac{1}{2\chi}\mathcal{R} - \frac{1}{\chi}\left(E^{\nu L}\left((-g)^{1/2}E^{\mu}_{L}\right)_{,\mu}\right)_{,\nu} = \frac{1}{2\chi}\mathcal{R} + \mathcal{L}',$$

and therefore according to the definition (14),

$$f^{\nu,}{}_{\alpha L}{}^{\mu}\left(-(-g)^{1/2}E^{\alpha L}\right)_{,\mu} = \frac{1}{\chi}E^{\nu}_{L}\left(-(-g)^{1/2}E^{\mu L}\right)_{,\mu},$$

so we deduce that

$$f^{\nu,}{}_{\alpha L}{}^{\mu} = \frac{1}{\chi}E^{\nu}_{L}\delta^{\mu}_{\alpha}.$$

[65] According to the definition (2)

$$\delta\mathcal{E}^{\alpha M} = c^{\alpha MIJ}\xi_{IJ} = \xi^{M}{}_{J}\mathcal{E}^{\alpha J},$$

so we deduce that

$$c^{\alpha MIJ} = \frac{1}{2}\left(\mathcal{E}^{\alpha J}\eta^{MI} - \mathcal{E}^{\alpha I}\eta^{MJ}\right) = \frac{1}{2}(-g)^{1/2}\left(E^{\alpha I}\eta^{JM} - E^{\alpha J}\eta^{IM}\right).$$

[66] It is simplest to calculate the variation of $\mathcal{L}'$ directly and then use (71), or equivalently (72), to read off the expression for $\mathcal{I}^{\nu IJ}$.

We find that since

$$\mathcal{L}' = \frac{1}{\chi}\left(E^{\nu L}\left(E^{\mu}\left((-g)^{1/2}\right)_{,\mu} + E^{\mu}_{L,\mu}(-g)^{1/2}\right)\right)_{,\nu} = \frac{1}{\chi}\left(g^{\nu\mu}\left((-g)^{1/2}\right)_{,\mu} + E^{\nu L}E^{\mu}_{L,\mu}(-g)^{1/2}\right)_{,\nu},$$

$$\delta\mathcal{L}' = \frac{1}{\chi}(-g)^{1/2}\left(\delta E^{\nu L}E^{\mu}_{L,\mu} + \left(E^{\nu L}(\delta E^{\mu}_{L})_{,\mu}\right)\right)_{,\nu} = \frac{1}{\chi}\left((-g)^{1/2}\left(\xi^{LM}E^{\nu}_{M}E^{\mu}_{L,\mu} + E^{\nu}_{L}(\xi^{LM}E^{\mu}_{M})_{,\mu}\right)\right)_{,\nu}$$

$$= \frac{1}{\chi}\left(\xi^{LM}\left((-g)^{1/2}E^{\nu}_{M}E^{\mu}_{L}\right)_{,\mu}\right)_{,\nu}$$

Thus we deduce from (72) that

$$\mathcal{I}^{\nu\mathcal{I}\mathcal{J}} = \frac{2}{\chi}\left((-g)^{1/2}E^{\nu[I}E^{J]\mu}\right)_{,\mu}$$

[67] According to the definition (2)

$$\delta e_{\alpha M} = c_{\alpha M}{}^{IJ}\xi_{IJ} = \xi_{M}{}^{J}e_{\alpha J},$$

so we deduce that

$$c_{\alpha M}{}^{IJ} = e^{[J}_{\alpha}\delta^{I]}_{M}.$$

[68]

$$c_{\psi}{}^{IJ} = -\frac{1}{4}\Gamma^{I}\Gamma^{J}\psi.$$

[69]
$$\mathcal{F}^{IJ} = \mathcal{P}^{\alpha M} {c_{\alpha M}}^{IJ} + \mathcal{P}^{\psi} {c_{\psi}}^{IJ} = \mathcal{P}^{\alpha [I} e_{\alpha}^{J]} - i\hbar c \frac{1}{4} \bar{\psi} E_L^0 \Gamma^L \Gamma^I \Gamma^J \psi.$$

This is the correct generator of local Lorentz transformations.

[70]
$$\mathcal{T}^{I\nu} = \frac{\delta \mathcal{W}}{\delta e_{I\nu}}.$$

[71] This is just the coefficient of ξ_{IK} in $\delta\mathcal{W} = \frac{\delta\mathcal{W}}{\delta e_{J\nu}} \delta e_{J\nu} = \frac{\delta\mathcal{W}}{\delta\psi}\delta\psi + \frac{\delta\mathcal{W}}{\delta\psi^*}\delta\psi^*$.

[72]
$$\mathcal{T}^{I\mu} e_{\nu}^{J} - \mathcal{T}^{J\mu} e_{\nu}^{I} = 0.$$

[73] The Rosenfeld expression should actually be $\mathcal{T}''_{ik} = e_k \mathcal{T}_i{}^{\nu} h_{k,\nu}$. Transcribing in modern notation this is

$$\mathcal{T}''^{IK} = \mathcal{T}^{I\mu} e_{\mu}^{K}.$$

[74]
$$\mathcal{T}'^{I}{}_{\nu} = \frac{\delta \Re \mathcal{W}}{\delta E_I^{\nu}} = -e_{\nu}^{J} e_{\rho}^{I} \frac{\delta \Re \mathcal{W}}{\delta e_{\rho}^{J}} = -e_{\nu}^{J} e_{\rho}^{I} \mathcal{T}_J{}^{\rho} =: (-g)^{1/2} T'^{I}_{\nu}.$$

[75] First note that
$$\frac{\partial (-g)^{1/2}}{\partial E_I^{\nu}} = -(-g)^{1/2} e_{\nu}^{I}.$$

Then referring to [47]

$$\frac{\delta \Re \mathcal{W}}{\delta E_I^{\nu}} = -e_{\nu}^{I} \Re \mathcal{W} + \Re i\hbar c (-g)^{1/2} \overline{\psi} \Gamma^I \left(\frac{\partial}{\partial x^{\nu}} + \Omega_{\nu} - i \frac{e}{\hbar c} \phi_{\nu} \right) \psi + \Re i\hbar c (-g)^{1/2} \overline{\psi} E_L^{\mu} \Gamma^L \frac{\delta \Omega_{\mu}}{\delta E_I^{\nu}},$$

and

$$T'^{I}{}_{\nu} = -e_{\nu}^{I} \Re W + \Re i\hbar c \overline{\psi} \Gamma^I \left(\frac{\partial}{\partial x^{\nu}} + \Omega_{\nu} - i \frac{e}{\hbar c} \phi_{\nu} \right) \psi + \Re i\hbar c \overline{\psi} E_L^{\mu} \Gamma^L \frac{\delta \Omega_{\mu}}{\delta E_I^{\nu}},$$

where $\mathcal{W} =: (-g)^{1/2} W$.

[76] $E_I^{\nu} = \delta_I^{\nu}$.

[77]
$$T'^{I}{}_{\nu} = -\delta_{\nu}^{I} W + i\hbar c \overline{\psi} \Gamma^I \left(\frac{\partial}{\partial x^{\nu}} - i \frac{e}{\hbar c} \phi_{\nu} \right) \psi,$$

where
$$W = i\hbar c \overline{\psi} \Gamma^{\mu} \left(\frac{\partial}{\partial x^{\mu}} - i \frac{e}{\hbar c} \phi_{\mu} \right) \psi + mc^2 \overline{\psi} \psi$$

[78]

$$T'^{00} = -i\hbar c\overline{\psi}\Gamma^a\left(\frac{\partial}{\partial x^a} - i\frac{e}{\hbar c}\phi_\mu\right)\psi - mc^2\overline{\psi}\psi.$$

As noted in [46], the mass term appears with the wrong sign in Rosenfeld's matter Lagrangian. The corrected expression is therefore

$$T'^{00} = -i\hbar c\overline{\psi}\Gamma^a\left(\frac{\partial}{\partial x^a} - i\frac{e}{\hbar c}\phi_\mu\right)\psi + mc^2\overline{\psi}\psi.$$

[79] Rosenfeld is now interpreting the energy density as arising from a single particle expectation value, where the ψ are conceived as elements of an L^2 Hilbert space. Thus he interprets

$$T'^{00}/c =< \psi|H|\psi >,$$

with the energy operator

$$H = \frac{\hbar}{i}\Gamma^0\Gamma^a\frac{\partial}{\partial x^a} - e\Gamma^0\Gamma^\mu\phi_\mu + \Gamma^0 mc.$$

The Dirac equation is then $H\psi = i\hbar\frac{\partial\psi}{\partial x^0}$.

[80]

$$T'^0{}_a = \Re i\hbar c\psi^*\psi_{,a} - e\psi^*\psi\phi_a.$$

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