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# COMPOUNDING RATIOS AND INTERVALS: AN EDUCATIONAL/HISTORICAL APPROACH IN MATHEMATICS AND MUSIC

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## INTRODUCTION

In this study I shall consider educational aspects of the development of ratio and proportion, focusing on the arithmetization undergone by these concepts in the light of the relations between mathematics and music. Since such relations, even if confined to the context of ratio and proportion, are fairly wide-reaching and also that the process of arithmetization is quite complex, we shall concentrate mainly on the instructional aspects of a structural peculiarity presented in such a fascinating dynamics. This peculiarity is the so-called *compounding ratios*, a curious feature present in the structure of ratio since the Classical Period whose irregular transformation into the operator multiplication is quite representative of the importance of theoretical music in the arithmetization of ratios. As a consequence we shall also point out features of the differences between *identity and proportion*, which are capable of being didactically explored with a mathematical-musical approach.

The reason for choosing music for the present approach is not only historical, but more specifically didactic insofar as the subtle semantic differences between *compounding* and *multiplication* and also between *identity* and *proportion* are clearer if one thinks of ratios as musical intervals when looking at such constructs. Grattan-Guinness (Grattan-Guinness, 1999, p.11) argues that the well-known difficulties in teaching fractions can be alleviated by converting the latter into ratios, and thus using a musical approach. These considerations corroborate the need to explore didactically specific contexts in which differences between given constructs manifest themselves more clearly.

This approach is also historical: the Classical Greek practice of manipulating ratios, predominantly performed up to the 14th century (Katz, 1993, p.291), belonged to an important tradition in the treatment of ratios, which is capable both of opening the minds of students to the notion of analogical structures underlying concepts concerning apparently different fields and of inviting them to put themselves in the place of the scholars who created and practiced such a tradition. It thus promotes an understanding of the scientific structures in the light of which certain mathematical concepts were, were handled and thus, too, an understanding of the apparently senseless way in which such mathematical concepts were manipulated for a long time before reaching the today's form. An awareness of these practices facilitates the acquisition of a flexible attitude concerning previous structures when confronting new problems, an essential tool for the resolution of problems and for creativity in mathematics.

The present approach also helps to reveal, by means of simple concepts such as ratio and proportion, the epistemological process often involved in the construction of mathematical theories, i.e. that of initially borrowing the structures of pre-existent analogical theories that then develop autonomously in their new context and adapt themselves to the practical problems with which the new theories come to grips in the course of their development.

In order to fulfill the aforementioned aim we shall first of all introduce some historical aspects of ratio in mathematical-musical contexts as well as of the corresponding structure in which compounding makes sense, and then follow these with examples of the practice of compounding on the monochord and by the didactic-epistemological aspects that underlie such a practice.

## Historic considerations: compounding ratios or musical intervals?

Mathematics and music have deep links already known since Antiquity. In the so-called experiment of the monochord, Pythagoras did not just establish correspondence between musical intervals and ratios of a string, but connected musical consonances to simple ratios—octave: $1:1/2$, fifth: $1:2/3$, fourth: $1:3/4$. Pythagoras' discovery through the monochord experiment casts light on a large number of discussions about musical theory that have ratios as their main characteristic. It is quite probable that, for cultural reasons, the Greek mathematicians, along with his contemporaries and predecessors, conceived of the theory of ratio as a generalization of music, inasmuch as the proprieties of strings and comparisons between pitches, as well as calculations related to such magnitudes through ratio and proportion, were an important part of mathematics from the Pythagoreans until Euclid (Abdounur, 2001, p.8; Grattan-Guinness, 1996, p.367).

This raises questions concerning the mathematical theories underlying the manipulation of ratios from Antiquity until the late Middle Ages, especially in musical contexts. The influence of both theoretical and practical problems confronted by music throughout its history are of great importance for the epistemological awareness of the history of ratio in dynamics of mathematical education, an awareness which can be useful for instance in grasping differences between basic albeit misunderstood concepts resulting from the definition of ratio, such as those that exist between *compounding* and *multiplication*, *identity* and *proportion* among others, differences which are hard or impossible to notice when these concepts are approached for instance only in arithmetical contexts.

There are several themes on the relation between mathematics and music or even between ratios and musical intervals which can be explored in mathematics education. We will concentrate here on an intriguing characteristic of the structure originally associated with the concept of ratio, namely compounding ratios, which we could call an operator, although it never attained the status of a technical term in mathematics (Sylla, 1984, p.19). Such an operator occurred tacitly in contexts involving ratios since the Classical period up to the 17th century, being eventually superseded by multiplication.

The structural change is from conceptions of operations—*compounding ratios*—strongly tied to contiguous musical intervals to theories that admit the composition of general ratios—multiplication—with an essentially arithmetic character, for example, the idea that a ratio is equal to a number. The point is how to approach in classroom dynamics an epistemological change such as this, which occurred in the course of the development of ratio, in such a way that one succeeds in creating an ordinary situation in which such a difference manifests itself more clearly than it does in purely arithmetical domains.

When one considers that this transitory structure with which ratios were very partially and irregularly equipped over a long period of their history is derived from musical contexts and also that *compounding* makes no sense out of musical contexts, it is quite reasonable to take music as the scenario for approaching such differences, since here the previous structure attached to ratio stands out. But before moving on to the instructional aspects of such a topic, we will have to delve into compounding ratios in more detail.

Some indicators of the different theories attached to the concept of ratio are found in connection with issues such as Euclid's restriction on the operation of *composition with ratios* implied in definitions 9 and 10, Book V as well as in proposition 23, Book VI (Heath, 1956, p.248). Such operations consisted of compounding ratios of the type $a{:}b$ with $b{:}c$ to produce $a{:}c$, which then allows the repetition of this process with $c{:}d$ and so on (Abdounur, 2001, p.5).

This operation, which had strong musical affinities, required in general that given a sequence of ratios to be compounded the second term of a ratio should equal the first term of the next ratio. Mathematically speaking, there is no reason to define this operation in such a way and we would not so define it unless we first observed its significance from a musical point of view, which understands what is otherwise a purely mathematical phenomenon as the adjoining of contiguous intervals. For instance, $(2{:}3).(3{:}4){::}(1{:}2)$ is structurally equivalent to the musical combination of the interval of a fifth with that of a fourth in order to generate an octave. Now, Pythagoras' Experiment seems to inform us of two things, whose didactical-epistemological implications we will try to point out later on. The first and more general point it makes is that mathematical ratios underlie musical intervals. But it also tells us more specifically that the *compounding ratios* underlie the *composition of musical intervals*, and even that, *due* to this link, composition of ratios in a Euclidean fashion is handled in this way. Quite apart from the interest which it holds for the historian of science, this ontological difference deserves attention in educational contexts.

We will try to propose now how to explore in didactic-pedagogical contexts these two completely different understandings of ratio, one geometric-musical where ratio has no semantic proximity with number and the other, where ratio is semantically a number, capable of being multiplied in the same way as numbers are multiplied. In order to emphasize such an important epistemological change present in the history of ratio, we will make use of musical contexts.

## PRACTICING MATHEMATICS/MUSIC: COMPOUNDING RATIOS/INTERVALS ON THE MONOCHORD

The problems described below were applied in workshops in mathematics/music carried out in São Paulo. The workshops comprised activities that reproduce, directly or analogically, meanings involved simultaneously in mathematics and in music. They were more concerned with the creation of circumstances that favor experiences of similarities between schemes behind the original and the reconstructed situations, than with the mere denotative reproduction of the former situation.

Compounding on the monochord is a case in point. *Compounding* in Euclid's sense must definitely *not* be put in the same category as *multiplication* although the former presents structural similarities with the latter. Both differences and similarities between *compounding* and *multiplication* concerned with musical and arithmetical fields respectively can be better felt and grasped with the help of an enriched reconstruction in learning/teaching context of the monochord's experiment. Such reconstruction can encourage students with promising tendencies in music to get interested in mathematics and vice-versa. Such crossing capacity not only stimulates the relationship between both *areas* and the related *skills* but also demands mathematics *skills* in musical *contexts* and musical *skills* in mathematical *contexts* through an simple arrangement involving elementary concepts.

Concerning the pertinent part of the workshop, monochords were first handed out to the participants who were initiated into the perception of basic musical concepts, such as musical interval, necessary for the following performance. Once the students discovered by means of the monochord the ratios $1{:}2$, $2{:}3$ and $3{:}4$ underlying the basic Greek consonances octave, fifth and fourth, respectively, one can set problems like:

- Let $L$ be the length which produce a determined pitch in the monochord. What is the length necessary to produce a pitch obtained raising the original one by an octave and a fifth, following by the lowering of two fourths? Listen to the resulting pitch in the monochord and compare that with the pitch obtained on the piano. Comment.
- Let $do$ be the pitch corresponding to the length $L$. Which is the pitch provided by the length $32L/27$? Indicate in terms of superposition of fourths, fifths and octaves, the successive steps to reach that result. In raising a fourth from the given pitch, what are the pitch and length obtained? Listen to resulting pitch in the monochord comparing it with the pitch obtained on the piano.

Such problems in particular, presented in a workshop with children between 11 and 14 years old in *Estação Ciência*—a museum for dissemination of science, culture and technology within the University of São Paulo —, for instance, demanded simultaneously musical and mathematical aptitudes and/or at least could awaken curiosity of students who were at first interested exclusively in either mathematics or music. Depending on where each student's greatest potential lies, students solve these kind of problems either by finding the interval and checking the compounding ratios which provide it or by finding the combination of ratios that when compounded provide the requested interval, and checking the interval.

Such problems provide one with the opportunity not only to experience, perhaps even unconsciously, the compounding of ratios but also to simulate operations with ratios in Greek and medieval musical contexts, inasmuch as the students have as basic operational elements the perfect consonances, that is, the discrete ratios $1:2$, $2:3$ and $3:4$, which in this context have no categorical relation with numbers in principle, but are merely instruments for comparison.

In order to illustrate my points, it may be worthwhile to describe some of the reactions that occur in solving these problems. I will take as an example a workshop for students of the ‘8th serie’—around 14 years old—carried out at ‘Escola de Aplicação’ in São Paulo. Because of size limitations, I will confine my discussion to some approaches to the first problem as well as some questions which were raised as a consequence. In this case, the solutions passed basically from a geometric approach to an arithmetic one.

First of all, the students were familiarized in the workshops with intervals and compounding of musical intervals/ratios in the monochord. This experience enabled them to compound contiguous intervals or mathematical ratios where the endpoint of the second magnitude of the first ratio coincided with the first magnitude of the second ratio—ratios of the type $a:b$ with $b:c$—which is what they saw in the monochord during the familiarization. The classroom was then divided into groups comprising students of different tendencies in order not only to make possible different kinds of interpretations of the problems but also to provide an appreciation of the diversified potential of each group since all problems would eventually claim the use of at least music and mathematics skills.

Initially, they were asked to solve problem one using a ruler with only four divisions and a compass. After visualizing how compounding operated in the monochord, students evinced basically two tendencies in solving the problem: one tendency was to make the calculation by always transferring the ratios to the string and dividing the string into as many parts as the denominator and then taking the number of parts that were in the numerator—in the case of $2:3$ two parts of the strings previously divided in 3 parts—which is clearly compounding in the classic sense. Other students tried to find the resulting note—in the case a la—but tried to check such a result by compounding the ratios $1:2$, $2:3$ and decompounding the ratios $3:4$ two times, as in the first case. In order to perform this operation they availed themselves of the operation, used in the first step-by-step demonstrations, of the basic consonance—octave, $1:2$; fifth, $2:3$; and fourth, $3:4$. In general, they found the part of the string which when sounded resulted in the note la without knowing precisely to which ratio or note such a point or pitch corresponded.

In this first stage, no arithmetical interpretations resulted. They did the procedure as in the demonstration of the consonances, in which we used rule and compass to build similar triangles in order to divide a segment in 2, 3 and 4 parts. The following question emerged:

- Do we get the same result if I change the order of the procedure?

They figured it out from a musical point of view, an approach which makes the answer fairly intuitive, since compounding is nothing but the ‘addition’ and ‘subtraction’ of intervals. Such an interpretation makes the commutativity of this operation more intuitive. It shows also to some extent how the musical context could facilitate the ‘feeling’ of the meaning of such a property in the structure of ratio.

The situation provided also a suitable context for moving on to the following question:

- How could we compound musical intervals when we know only the lengths of the strings whose ratio provide each interval? Again without metric ruler.

In this case some students tried to adapt by trial and error the first term of the second ratio to the second term of the first by taking ratios equivalent to the second term expressed as multiples of its two original magnitudes. A musical solution also emerged. For this, they tried to hear the intervals defined by each pair of strings by singing their compounding and sometimes keeping the partial result in a keyboard in order to keep the tuning. They confirmed the result doing it musically sometimes step by step, at other times at the end of the operation, based on the initial musical auditive experience with intervals and consonances. They could do it almost automatically, subsequently verifying the length of the string that corresponded to the discovered pitch. To accomplish such an operation they must always find the ‘musical’ fourth proportion insofar as in each step they have a reference ratio and the first factor of a second ratio that provided the lower note over which the reference interval should be translated.

Others students even tried a mixed solution by guessing through hearing the probable ratio from which they could give a good guess as to the factor by which it was necessary multiply both factors of the second ratio. In all cases the students often make use of a proportional pair of strings which are naturally not equal but that have some property which makes them similar in some way to the first pair. This feeling of similarity realizable by hearing is one important point that pervaded many different situations in these workshops and both emphasized and eventually eased the differentiation between proportionally and equality, a feeling which disappeared when they later faced the problem with an arithmetical approach using a metric ruler. The advantage of the musical approach in comparison with the geometrical one consists in the fact that the former provides the feeling, based on a perceptive skill, that both pairs of magnitudes are not equal but that at the same time they have a common attribute, which is musically the interval defined by them. In the face of such similar ratios/intervals, some comments like the following were heard:

- They are not equal but one is ‘as if’ it were the other.

The rationalization of such a feeling was refined when not only harmonic but also melodic versions of the same ratio were provided. Then some comments like the following one appeared:

- The notes ‘walked’ or ‘climbed’ with the same step:

They are probably doing albeit not necessarily consciously a musical or logarithmical approach.

In order to provide a similar visual perception by geometry, on the other hand, the four magnitudes should be laid in a particular configuration—not necessary in music —, which was also approached—as the following shows—in order to strengthen such a differentiation.

In such a dynamics, the following question came out.

- Could we calculate it only once?

Then similarity was introduced so that one could build precisely the proportional second ratio in such a way that its first term had the same measure of the second term of the first ratio, emphasizing a geometric/musical connotation to proportionality.

Still without metric rule, it was possible to pose the following question:

- Could we calculate the compounding of all ratios applying it at the end to the monochord?

One possibility was to do it analogically to the geometric procedure using now whole numbers, which involves the knowledge that $a:b :: ma:mb$—proposition 18 of Book VII of The Elements—going on working technically just with integers. In such a dynamics the following question came out:

- What do we do to compound $a:b$ with $c:d$ when there is no integer m so that $mc = a$?

When we dealt only with geometrical magnitudes this question did not arise, since one can always adapt one magnitude to another but that is not the case with whole numbers to be adapted to each other using integer multiples.

In this case, one must multiply the numerator and denominator of both ratios, resulting as factors $c$ and $b$ respectively which make the original compounding proportional to $(ac:bc).(bc:bd)::ac:bd$. Based to some extent on the trial and error experience done before with geometrical magnitudes, they tried now to do something analogical with integers represented geometrically which resulted eventually in the use of the Minimal Common Multiple between $b$ and $c$.

The compounding of all ratios was curiously very easily done with intervals, that is, from a determinate interval with a certain low pitch, they could build the correspondent equivalent interval—proportional ratio—from hearing and feeling the same ‘growth’ of interval.

The comments and questions mentioned above concerning the solution of the first problem reflect to some extent the dynamics of this workshop. The example mentioned above tried to reflect partially how the workshops could provide a suitable environment to experience this arithmetic sense of ratios, by introducing this approach before turning to the metric ruler.

The problem was repeated allowing the use of metric rule and gradually ratios and compounding were equated to decimal numbers and multiplication respectively, thus diminishing the emphasis in the differentiation between identity and proportionality.

It was possible to realize that the problem became even more interesting insofar as one could restrict the available tools for the solutions: compass, non-metric ruler, metric ruler, instruments —which provide different meanings to ratio and proportion, and could get the student to operate at times with compounding, and at other times with multiplication. Such an enriched arrangement proves useful not only for illustrating the importance of ratio as a medium for comparison but also and most importantly for providing a context for practicing the differentiation between both compounding and multiplication as well as between proportionally and identity within a meaningful practical situation.

## DIDACTIC-EPISTEMOLOGICAL ASPECTS

Besides the difference between *compounding* and *multiplication*, there are deeper differences within the arithmetization of ratios that become transparent through the aforementioned arrangement, such as that between *identity* and *proportion*. In Euclid, the idea of equality of ratios is not as natural as that of numbers or magnitudes. Such a way of establishing relations between ratios gains greater meaning when we consider that on the monochord, for instance, *do* - *sol* and *la* - *mi* are the same intervals—in this case, a fifth—but they are not equal, inasmuch as the latter is a sixth above the former, or even that do - sol ‘is as’ la - mi. The identity is normally a philosophically difficult concept to be worked out in learning/teaching dynamics. Such difficulty can be eased by stressing the distinction between *identity* and *proportion* in mathematical/musical contexts, where such differences become clearer when visible and ‘*audible*’.

The problems and the device mentioned above also encourage the perception of such a difference insofar as the students can hear the intervals provided by proportional ratios like $9:12$ and $12:16$—both are fourths, that is, the same intervals, but they are not equal—which are proportional but definitely *not identical*. This elucidates by the use of mathematics and music the differences and similarities between both concepts which also contribute to the better understanding of the identifications of *ratio* and *fraction* and of *proportion* and *equality.* It opens several possibilities for exploration of such concepts in both contexts. For instance, they can find the forth proportional and deduce what is the associated pitch or reciprocally, given an interval, they can figure out the note which will produce the same interval given a determinate lower pitch: both situations deal with proportional magnitudes in mathematical and musical contexts simultaneously. The students must not necessarily be aware of the epistemological procedure underlying such dynamics. What is actually important is that they experience such a situation and thus establish a reference with which they can bridge and anchor the comprehension of future situations involving these concepts. In the same way, the experience will enable them to detach concepts associated with fixed areas and interweave them in a more general context.

The aforementioned arrangement in teaching/learning as well as the long history of ratio and proportions show that, within the rich semantic field associated with these concepts, ratio was a natural vehicle for human beings to use in comparing different contexts through proportions, that is, analogies. In this sense, the proposition that $3:2$ corresponds to a fifth, as well as that one that the aforementioned intervals of fourths are proportional mean that these two concepts pertaining to mathematical and/or musical fields are capable of being compared to one another by means of the ratio of numbers and the interval between notes through proportions. In this sense, it is possible to experience that the geometrical/musical proposition $A:B::C:D$ is semantically distinct from yet structurally similar to the arithmetical proposition $A \div B = C \div D$, as well as that the corresponding cases in which ratios are not proportional and fractions are not equal.

Reciprocally, by means of the device of the monochord, ratio and proportions are viewed as instruments for evaluating the degree of similarities between different contexts. Such a device can also help the comprehension of the categorical distinction between ratio and proportion—sometimes misunderstood—inasmuch as ratio is clearly viewed as a *definition* involving two magnitudes of the same kind whereas proportion functions in all the aforementioned situations either as a *logical proposition* to which one may attribute a valuation or as a tool to make a proposition true. In the case, such a difference is experienced through the question about the plausibility of the equality between two intervals or of the proportion between two ratios. The differences between these two mathematical entities are less ambiguous when understood in this way than when viewed in purely arithmetical contexts.

## Conclusion

The present musical approach widens our comprehension of ratio and proportion in mathematics not only because of its historical-cultural contextualization and the interdisciplinary aspect which underlies it, but also, and most importantly, because of the role that analogical thought plays in the construction of meaning, in this case, that of ratio and proportion. If we wanted to extend Kieren's argument (Kieren, 1976, p.102) about rational numbers to ratios, we could claim that to understand the ideas of *ratios*, one must have adequate experience with their many interpretations. Throughout the history of mathematics and theoretical music, ratio and proportions assumed different meanings with discrete or continuous natures in regard to geometry, music and/or arithmetic. Among such meanings, ratio can be seen as a tool of comparison by means of proportions, a musical interval, a fraction, a number, an invariant with respect to proportion, a common thread between distinct contexts with regard to proportions whereas proportion can be seen as a vehicle to compare ratios, an equality, a relation, a function etc. The aforementioned device not only provides a fertile ground for the understanding of the subtle differences and structural similarities underlying the diversity of interpretations associated with ratio and proportions but also contributes to constructing and to experiencing in a broader way their associated meanings.

In a general sense, discovering common schemes and archetypes is an efficient way of constructing concepts that concern in principle different areas. An analogy or metaphor used in a sensible and discerning way may re-configure a student's thought in a problematic situation of learning, enabling a better understanding of matters that escape immediate intuition, or that seem too abstract to him/her, such as the many interpretations associated with ratio and proportions as well as with the wide variety of structures historically associated with them.

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