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Infinitesimals in Spain: Antonio Portuondo's *Ensayo sobre el Infinito*

INFINITESIMALS IN SPAIN: ANTONIO PORTUONDO'S *ENSAYO SOBRE EL INFINITO*

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Abstract

In this paper the authors study the work of the Spaniard Antonio Portuondo on the theory of infinitesimals during the revival of this theory in the decade 1870-1880. An analysis of the book *Ensayo sobre el infinito* is the guiding line linking Cauchy and Nonstandard Analysis.

1 Introducción

According to Lützen (Lützen, 2003), the route towards rigour of the new science of Mathematical Analysis originated during the second half of the 17^{th} Century. Many previous ideas from a dearth of authors were formalised in the work of Newton, whose idea of rigour was embodied in his inkling towards proofs in the Euclidean geometrical style. Newton's development of Calculus was based in a loose idea of the concept of limit, while, on the other hand, Leibniz and the Bernouillis developed their theory with a different style and their idea of infinitesimals -stemming from the concept of a finite increment- was that of actual quantities that could be used in computations but could be eliminated, according to definite rules, once the interesting results were obtained. This view was severely criticised by scholars like Berkeley (Berkeley, 1734) as lacking the necessary philosophical foundations. Moreover, the era of Euler is that of algebraisation of Mathematical Analysis; this amounts to saying that computational and algorithmic properties reigned over foundational quiestions. Euler's infinitesimals were actual zeroes and infinities subject to the ordinary rules of computation with some elementary precautions (Cuesta Dutari, 1985).

Cauchy's approach was based on the supposedly metaphysics-free idea of limit as the cornerstone of the presentation of Mathematical Analysis, a procedure that was to pervade Mathematical Analysis until the present day. In his *Cours d'Analyse*, Cauchy introduced infinitesimals as a particular class of variables, *i.e.* those whose values approach zero, and he also presented the basic rules for handling these quantities in mathematical practice. Nevertheless, Cauchy did not develop a general theory of infinitesimals, and some other authors (Duhamel, 1856; Freycinet, 1860) tried to fill the gap in their treatises. In fact, they came back to metaphysical ideas and concepts.

A new viewpoint was presented by Weiestrass when the idea came into scene of founding Mathematical Analysis on Arithmetic (Dugac, 1973). In his view, all considerations on the behaviour of infinitesimals were translated into relationships between (real) numbers and concepts like convergence of limits and continuity were formulated this way. Of course, Weierstrass' style is rigourous as far as Arithmetic is considered a sound Mathematical Science, and from the practitioners' standpoint, it amounts to obtain estimates and bounds, not always an easy task.

The formality of the Weierstrassian Analysis is in sharp contrast with the more flexible and operational, even more intuitive, way of handling the idea of infinitesimals on the only idea of some ordering properties. Therefore, around the last third of the 19^{th} century there was a revival of infinitesimal theories by authors like the French Duhamel, Freycinet, etc, and the German DuBois-Reymond (DuBois-Reymond, 1875, 1882).

The reception of Cauchy in Spain has been studied elsewhere (Pacheco, Pérez et Suarez, 2008), and books in the Cauchy tradition and style were published in Spain around 1880, already including developments posterior to Cauchy found in the most common French treatises. Nevertheless, up to the year 1900 there appeared books written -usually for the Military Schools (Vallejo, 1813-1832; Feliu, 1847-1848; Belón, 1876; Villafañe, 1892; Toro, 1894) - following the Lagrange formulation of Mathematical Analysis.

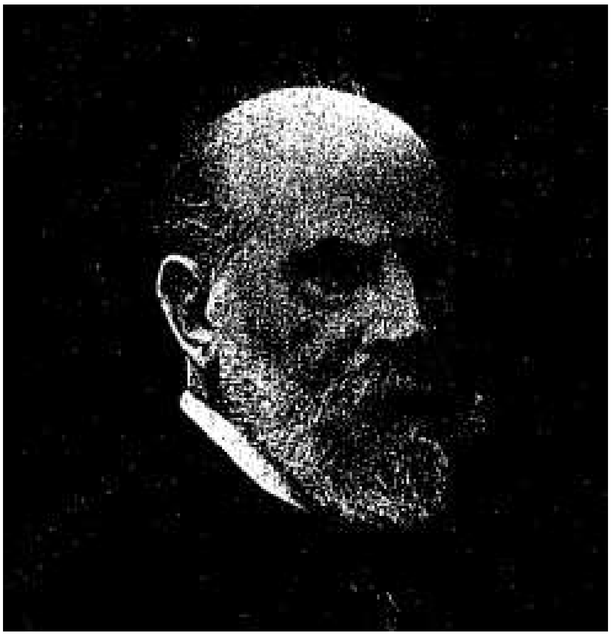


Figure 1: The Spanish engineer and mathematician Antonio Portuondo (1845-1927).

The revival of infinitesimals also had a Spanish contributor. In the year 1880 Antonio Portuondo, a civil engineer and mathematician, published his monograph *Ensayo sobre el infinito* (an Essay on Infinitesimals, *Ensayo* from now on) where he studied, to a certain depth, the original definitions of Cauchy. The aim of this paper is to comment on the ideas of Portuondo, to make a criticism thereof, and to find a possible connection with later developments such as Non-Standard Analysis.

2 Portuondo, his mathematical knowledge, and a first glimpse of *Ensayo*

As we have already pointed out, Antonio Portuondo (1845-1927) was a civil engineer of Cuban-Spanish origin who spent most of his life in Madrid. He seems to have been rather involved in the cultural and educational life at the Spanish capital. He was a close friend and supporter of Francisco Giner de los Ríos (1839-1915), one of the founders of the liberal school system "Institución Libre de Enseñanza". We have found his name as a member of the Ateneo de Madrid society between 1896 and 1904.

Figure 2: Portuondo's signature.

Portuondo was a Professor at the School of Civil Engineering (Ingenieros de Caminos, Canales y Puertos, the Spanish version of the French Ingénieur des Ponts et Chausées) from 1883 until his retirement around 1813, and was rather known for his 1879 joint translation with his brother Joaquín of the classical two-volume *Traité de Géometrie* by Rouché and Comberousse, as well as the author of a companion volume to this translation, *Notas al Tratado de Geometría de E. Rouché y C. Comberousse*. He was also the author (1912) of a series of articles published in the "Revista de Obras Públicas" (Portuondo, 1912b) under the title *Apuntes de Mecánica Social* (Essay on Social Mechanics), where an attempt is made towards the application of the laws of Mechanics to Sociology.This book is studied in a forthcoming paper, where the influences of other authors in Portuondo are considered. Most of the biographical data have been obtained from the obituary published by the "Revista de Obras Públicas" in 1927 (Machimbarrena, 1927).

He wrote some more books, or rather lecture notes, for the students at the Engineering School, such as *Discusiones de Geometría* (Geometrical Discussions), *Apuntes para la clase de Mecánica* (Lecture notes on Mechanics), and *Apuntes sobre Probabilidades, teoría de los errors y método de los mínimos cuadrados* (Lecture notes on Probability, Error Theory, and the Least-squares Method).

Ensayo is an isolated work unrelated to any previous writing by the author, but he was rather proud of it because he managed to publish an excerpt of it thirty two years later, in 1912, under the title *Les Lois Infinitésimales dans l'Analyse Mathématique* in the Revue Générale des Sciences Pures et Appliquées (Portuondo, 1912a). This long silence and such a late resurrection of the topic can be interpreted variously. An interesting approach could be the following: *Ensayo* appeared in 1880, a few years after DuBois-Reymond's articles on the subject (DuBois-Reymond, 1875), and a couple of years before the book by DuBois-Reymond *Die allgemeine Functionentheorie* was published. Then the subject disappeared from the mathematical scene until the 1910 book by Émile Borel, *Théorie de la Croissance*, and the first edition of Hardy's *Orders of Infinity: The Infinitärcalcül of DuBois-Reymond*, also from 1910. Quite possibly, Portuondo felt that he should reappear in connection with these publications. Portuondo's article does not add anything to the views expressed in 1880, and the lack of references in the article makes it difficult to assess whether Portuondo had read these newer publications. The thesis that Portuondo considered this book as his most interesting contribution to Mathematics is again supported, by the fact that, during his last years of life, he prepared a French translation, which appeared in Gauthier-Villars during 1927, a few months after his death.

Another complementary explanation would be the tendency, present in some 19^{th} century Spanish mathematicians to engage themselves in foundational research on some topic with the idea of making a fundamental and definitive contribution. Such is the case of Rey Heredia and the metaphysics of complex numbers (García de Galdeano, 1899; Pacheco, Pérez et Suarez, 2006). As a rule, these attempts yielded clumsy and complicated essays whose mathematical value is difficult to assess.

Ensayo is a small book in 8^{o} with 168 pages. It is divided into an short Introduction, a few pages of basic notions (*Nociones fundamentales*), three Chapters and a long Appendix, as well as a table in a dépliant between pages 38 and 39, and another dépliant sheet with beautiful line drawings at the end of the book. It was printed in Madrid the year 1880, where a review appeared the same year (Pardo, 1880), and did not have any further editions in Spain. As it has been explained, a French edition appeared shortly after the author's death, forty-seven years after the original one. The French edition does not add anything new to the original Spanish edition of 1880.

Figure 3: Spanish and French editions of Portuondo's book, 1880 and 1927.

The first chapter is devoted to the ordering and valuation of infinitesimal laws, the second one deals with how to compute with infinitesimals, and the third and last presents a series of theorems needed in the development of Calculus. The long appendix or NOTA (pages 126-158) contains an analysis of the possible relationships between any two infinitely small (large) functions and has a spirit very similar to DuBois-Reymond's article of 1875 (DuBois-Reymond, 1875).

The author writes in the Introduction [1]:

> "The infinitesimal Method, discovered by Newton and Leibniz, has been established along the present Century (...) with all the rigour that scientific spirits demand",

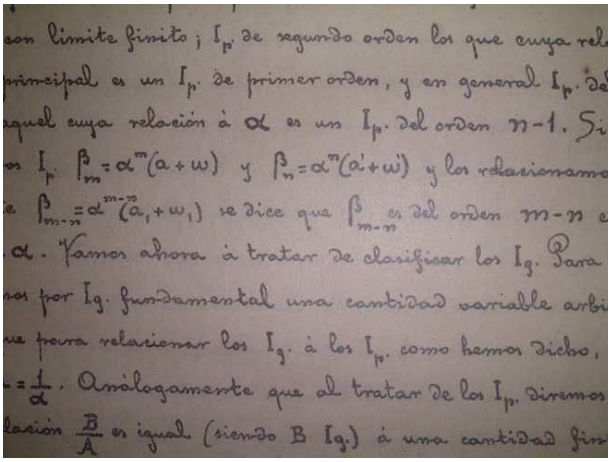
and he quotes Duhamel and his *Élements de Calcul Infinitésimal* as a rather satisfying book. Nevertheless, Portuondo adds:

> "to my knowledge, there is not yet a well-developed general and abstract theory of infinitesimals: Maybe it would clear up many confusing ideas and ban what in my opinion are but illusions of worried spirits (...) I offer some ideas in this little Essay, with the desire that more competent people will develop a complete Theory of Infinitesimals".[2]

[1] El Método infinitesimal, inventado por Leibniz y Newton, ha sido establecido en el siglo actual (...) con todo el rigor que pide el espíritu científico

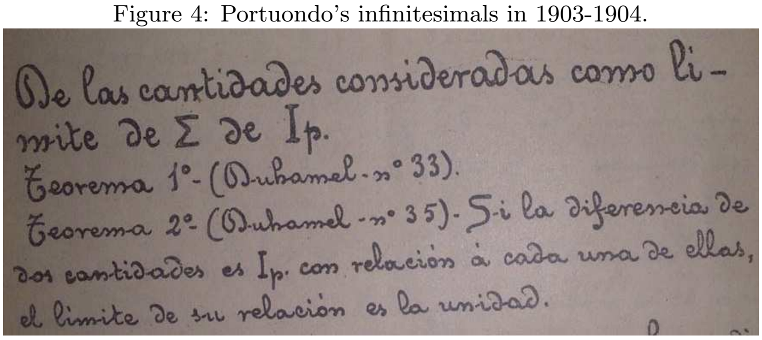
[2] No se ha dado cuerpo (que yo sepa) a una teoría general y abstracta de lo infinitesimal: tal vez ella haría imposible la confusión de ideas, y extirparía de raíz las que, en mi entender, son ilusiones de espíritus preocupados (...) apunto algunas ideas en este pequeño ensayo, con el deseo de que otros más competentes formen una Teoría completa de lo infinitesimal.

The ideas of Portuondo were used in the teaching of mathematics at the Escuela de Ingenieros de Caminos for a long time. Figures 4 and 5 show some anonymous lecture notes on Calculus in use during the years 1903-1904 at the School. The notation of Portuondo for infinitesimals is easily seen, as well as the quotations of Duhamel's Treatise as a source book.



con límite finito; I_p de segundo orden los que cuya rel
principal es un I_p. de primer orden, y en general I_p. de
aquel cuya relación à α es un I_p. del orden $n-1$. Si
os I_p. $\beta_m = \alpha^m(a+\omega)$ y $\beta_n = \alpha^n(a'+\omega')$ y los relacionamo
e $\beta_{m-n} = \alpha^{m-n}(a_1+\omega_1)$ se dice que β_{m-n} es del orden $m-n$ e
α. Vamos ahora à tratar de clasificar los I_g. Para
os por I_g. fundamental una cantidad variable arbi
ue para relacionar los I_g. à los I_p. como hemos dicho,
$= \frac{1}{\alpha}$. Análogamente que al tratar de los I_p. diremos
lación $\frac{B}{A}$ es igual (siendo B I_g.) à una cantidad fin

Figure 4: Portuondo's infinitesimals in 1903-1904.



De las cantidades consideradas como li-
mite de Σ de Ip.
Teorema 1°- (Duhamel-n° 33).
Teorema 2°- (Duhamel-n° 35)- Si la diferencia de
dos cantidades es Ip. con relación à cada una de ellas,
el límite de su relación es la unidad.

Figure 5. Introduction to integration. Duhamel's Treatise as source book.

3 Mathematical background of *Ensayo*

Cauchy's approach towards a rigorous foundation of Mathematical Analysis, specially on the matter of consistency, is described in the famous 1821 *Cours d'Analyse de l'École Polytechnique.* According to this text, any infinitely small quantity can be written in terms of an arbitrarily chosen infinitely small α as $k\alpha^n(1 \pm \varepsilon)$, where $k \neq 0$ is finite quantity, n is a whole number called *the order* of the infinitesimal and $\varepsilon = o(\alpha)$, in Landau's notation. The following quotation shows that Cauchy tries both to be cautious and to free Mathematical Analysis from objections like those of Berkeley (Cauchy, 1843, p. 16):

> Pour écarter complètement l'idée que les formules employées dans le Calcul différentiel sont des formules approximatives et non des formules rigoureusement exactes, il me parait important de considérer les différentielles comme des quantités finies, en les distinguant soigneusement des accroissements infiniment petits des variables. La considération de ces derniers accroissements peut et doit être employée, comme moyen de découverte ou de démonstration, dans la recherche des formules ou dans l'établissement des théorèmes. Mais alors le calculateur se sert des infiniment petits comme d'intermédiaires qui doivent le conduire à la connaissance des relations qui subsistent entre des quantités finies, et jamais, à mon avis, des quantités infiniment petites ne doivent être admises dans les équations finales, où leur présence deviendrait sans objet et sans utilité.

The next source for *Ensayo* is the 1856 *Traité* of Duhamel. Jean-Marie Duhamel (1797-1872) was the author of many pedagogical works, and some of them were most influential in the development of Spanish Mathematics, as shown by the books of Archilla (Archilla y Espejo, 1880), Portuondo (Portuondo, 1880), and later ones like that of Pérez de Muñoz (Pérez de Muñoz, 1914). Duhamel's books were employed as textbooks in many places in Spain (cf. (García de Galdeano, 1899; Vea Muniesa, 1995; Velamazán, 1994)). The ideas of Duhamel on the foundations of Mathematical Analysis pointed towards establishing the Differential and Integral Calculi on infinitesimal methods (Duhamel, 1856, T.1 p. XX):

> Ainsi, dans cet essai, que nous espérons rendre un jour moins imparfait, notre objet a été l'étude de la méthode infinitésimale considérée en elle-même; et les procédés si importants du calcul différentiel et du calcul inverse, ont été les moyens d'exécution des opérations auxquelles cette méthode a ramené la solution des questions qu'elle s'est proposées.

With this aim Duhamel works on the ideas of Cauchy and presents the principle of "equivalent infinitesimals" (Duhamel, 1856, T.1 pp. 35-38) and the principle of substitution of infinitesimal quantities:

> Lorsque la limite du rapport de deux quantités est l'unité, leur différence est infiniment petite par rapport à l'une quelconque des deux; *c'est-à-dire que* le rapport de cette différence á l'une quelconque de ces quantités a pour limite zéro. (...) si la différence de deux quantités est infiniment petite par rapport à l'une d'elles, la limite de leur rapport est l'unité (...) *La limite du rapport de deux infiniment petites n'est pas changée, quand on le remplace respectivement par d'autres qui en différent respectivement de quantités infiniment petites par rapport à eux.*

Jules Hoüel (1823-1886) wrote (Hoüel, 1878, T.1 p. VIII) in his 1878 *Cours de Calcul Infinitésimal*:

> On doit à Duhamel d'avoir, le premier, formulé nettement le principe qui identifie ces méthodes, si diverses qu'elles soient en apparence. Le but du Calcul infinitésimal est généralement la détermination des limites de rapports ou de sommes de certaines variables auxiliaires, appelées *quantités infiniment petites*, et le plus souvent ce but ne peut être atteint qu'en remplaçant ces variables par d'autres quantités susceptibles d'une expression plus simple, et conduisant au même résultat final. Le principe de Duhamel, que l'on peut nommer *le principe de substitution des infiniment petits*, consiste en ce que, dans les deux cas cités, on peut remplacer un infiniment petit par un autre infiniment petit dont le rapport au premier ait pour limite l'unité. En ne perdant jamais de vue ce principe, on pourra se servir en toute sécurité du langage et de la notation des infiniment petits, qui a sur celui de la méthode dite *des limites* l'immense avantage de la concision et de la simplicité, et qui seul permet au géomètre de se laisser guider, comme moyen d'intuition, par le sentiment de l'évidence tiré de la considération des grandeurs finies.

Thus Duhamel was able to affirm that:

> Ainsi toute suppression ou altération quelconque qui n'a d'autre effet que de faire négliger des termes infiniment petits dans une équation finale dont les deux membres ont des limites finies, peut être faite sans aucune erreur' dans les résultats.

Four years after the *Traité* by Duhamel, Charles Freycinet (1828-1923) published *L'Analyse Infinitésimale. Étude sur la Metaphysique du haut Calcul*, where a sharp criticism is made on the procedures employed in the development of Mathematical Analysis (Freycinet, 1860, p. IX):

> J'ai remarqué qu'une première étude de l'Analyse infinitésimale laissait dans l'esprit beaucoup d'incertitude et d'obscurité. On ne pénètre pas immédiatement la métaphysique de cette science, et l'on ne se rend pas compte de ce qui assure la rigueur des résultats à travers l'apparente inexactitude des procédés.

And he carries on like this:

> Tandis que dans la Géométrie et l'Algèbre ordinaires on raisonne sur des quantités toujours finies et déterminées, dans l'Analyse infinitésimale, au contraire, on abandonne les éléments véritables pour considérer des quantités auxiliaires, d'une petitesse indéfinie, et dont les dimensions ne sont jamais arrêtées, ce qui est bien propre à laisser une impression vague et incertaine. Non-seulement ces infiniment petits demeurent indécis, mais les relations établies à leur sujet ne semblent pas parfaitement rigoureuses, car on opère à tout instant comme s'ils pouvaient être remplacés par d'autres qui en diffèrent réellement. Enfin ils disparaissent toujours des formules où l'on avait débuté par les introduire, en sorte qu'on est en droit de se demander si l'on néglige ce qui a de la valeur, ou si l'on s'était servi de ce qui n'en avait pas. Aussi, tout en acceptant des résultats d'une incontestable exactitude, on ne se sent pas satisfait de la voie suivie pour les obtenir. C'est assurément un' grave défaut logique : il ne suffit pas d'atteindre le but, il faut encore savoir de quelle manière on y arrive.

Freycinet emphasises on the logical foundation of the concept of variable. He considers that such a quantity must be always variable: Therefore it cannot equal its limit (Freycinet, 1860, p. 21):

> Le propre de la limite et ce qui fait que la variable ne l'atteint jamais exactement, c'est *d'avoir une définition autre que celle de la variable*; et la variable de son côté, tout en approchant de plus en plus de la limite, *ne doit jamais cesser de satisfaire á sa définition primitive.*

On the other hand, infinitesimals are also variables, thus(Freycinet, 1860, p. 23):

> L'infiniment petit n'est pas une quantité *très-petite*, ayant une valeur actuelle, susceptible de détermination. Son caractère est d'être éminemment variable et de pouvoir prendre une valeur moindre que toutes celles qu'on voudrait préciser. Il serait beaucoup mieux nommé *indéfiniment petit*; mais la première appellation ayant prévalu dans l'usage, nous avons cru devoir la conserver.

On how to substitute or eliminate infinitesimals, Freycinet writes (Freycinet, 1860, p.166):

> La conclusion générale à tirer des réflexions précédentes, c'est que les équations infinitésimales sont toujours amenées, par des suppressions légitimes, à ne renfermer que des quantités *du même ordre de grandeur*; puisque toutes celles d'ordres supérieurs disparaîtraient nécessairement dans le passage aux limites. On doit voir là une sorte de principe d'homogénéité analogue à celui, d'après lequel, en algèbre, les termes d'une équation exprimant la loi analytique d'un phénomène quelconque, sont tous du même degré.

To conclude, Freycinet states, after quoting the *Réflexions sur la Métaphysique du Calcul Infinitésimal* by Carnot (1813), the *Théorie des Fonctions Analytiques* and the *Léçons sur le Calcul des Fonctions* by Lagrange (1797, 1806), states that: *"On peut dire que la méthode des limites avait perdu en simplicité ce qu'elle avait gagné sur sa rivale"*.

A remarkable aspect in *Ensayo* is the scarcity of refererences to other authors. Only Sturm is quoted on the sum of infinitely many infinitesimal quantities (footnote in p. 109), and a general citation of the Treatises of Bertrand, Cournot, Duhamel, Sturm (footnote in p. 126) are to be found in the book. Contemporary authors on the same topic, like Paul DuBois-Reymond (1831-1889) and Giuseppe Veronese (1854-1917) are missing.

4 A study of Ensayo

4.1 The body of the book

Ensayo starts with a 14-page Introduction where a rhetorical definition (Portuondo, 1880, p. 3) of a continuous variation law is found, much in the line of Freycinet:

> In Higher Analysis not only constant magnitudes do appear, but also variable ones. It should be understood that by variable we understand the continuous law of variation according to which the magnitude proceeds: Not only can the quantity take various values, it is actually variable, in such a way that it takes all intermediate values between any two given ones. I emphasise: Higher Analysis deals with continuous laws of variation. In order to be well defined, the law of variation of a particular quantity must be dependent on another quantity (or on several ones), whose otherwise non-defined continuous formation be simply continuous. For these ones the law is not conceivable, for its continuous variation is isolated and independent. One could say that the first one is a function of the second (ones), which might be called independent variables.[3]

Portuondo proceeds (Portuondo, 1880, p. 9-10) with the definition of a limit:

> A constant quantity towards which the variable approaches INDEFINITELY is called the limit of the variable"[4]

The main feature of the definition is that it allows oscillations in the approaching variable around the limit, but in order to comply with Cauchy's rule that the variable never attains the limiting value, the possible cases of equality are simply banned by being excluded from the values taken by the variable. Portuondo is not aware that this procedure actually breaks the supposed continuous variation of the quantity.

Next (Portuondo, 1880, p. 11) the uniqueness of the limiting values is pointed out[5], and the author proceeds (Portuondo, 1880, p. 11-12) with the definition of infinitely small and infinitely large quantities:

[3]En el Análisis superior no intervienen tan sólo magnitudes constantes, sino también la *variables*; pero entiéndase que cuando se habla de magnitudes variables, lo que se da, lo que se busca, o en general lo que se considera es la *ley de variación* o formación continua de magnitudes por la cual procede la cantidad: esta no sólo es susceptible de tener diversos valores, sino que es *actualmente* variable, de tal manera, que pasa por todos los valores intermedios entre dos cualesquiera de ellos. Lo repito: lo que el Análisis superior considera es la *ley de variación continua*. Esta Ley de variación continua de una *determinada* cantidad, requiere (para ser algo definido y pensado) la dependencia de otra u otras cantidades, cuya formación no definible sea simplemente continua; en éstas no se concibe *ley*, pues su variación continua es independiente y aislada: se dice que la primera es *función* de la segunda o de las segundas, que se llaman *variables independientes*.

[4]Se llama *límite* de una variable *a una cantidad constante a la cual se aproxima* INDEFINIDAMENTE *la variable.*.

[5]Una cantidad variable no puede tener más que un límite, porque es absurdo pensar en que sus valores sucesivos puedan diferir tan poco como se quiera de dos constantes desiguales: de aquí que dos expresiones distintas de una misma *ley de variación*, ó (como se dice ordinariamente) dos cantidades variables constantemente iguales tienden hacia un mismo límite (por las variaciones continuas de sus respectivas variables independientes), o en otros términos, sus límites son iguales (como se dice ordinariamente). En esto consiste el *Principio de los límites.*

> Variables that grow indefinitely are called infinitely large ones, and those indefinitely decreasing are called infinitely small. Both would be more properly called indefinitely large (resp. small) ones. The Calculus or Analysis where this sort of variables are employed is called Infinitesimal Calculus or Analysis.[6]

Portuondo does not explicitly use the absolute value of differences, although it is clear from his explanations that he refers to what Cauchy had called 'the numerical value', i.e. the absolute value. Therefore he is telling us that the words 'grow indefinitely' and 'decrease indefinitely' actually refer to the absolute value of the quantities involved.

After these general ideas the basic concept for the logical building of the theory is introduced, that of the complementary variable (Portuondo, 1880, p. 11-12):

> From the definition of a limit one sees that whenever a variable tends to a non-zero limit, the difference between the variable and the limit is an infinitely small quantity; this complementary infinitely small quantity will depend on the same variables as the variable with limit, and both vary simultaneously, in such a way that when the first one tends to its limit, the second one, which is -so to say- its complement, decreases indefinitely...[7]

Now we see that Portuondo's realm is that of continuous functions having either a finite or infinite limit, i.e. $\{\varphi(x)|\lim\varphi(x)=a, \quad a\in\mathbb{R}\cup\{\pm\infty\}\}$[8]. According to the definition of the complementary variable, from today's viewpoint Portuondo is reducing his study to that of functions having zero limit, but things are not so simple: following Freycinet our author states (Portuondo, 1880, p. 12) that:

> Infinitely small or infinitely large quantities do not have limits. Zero, considered as a would-be limit of infinitely small quantities, has no sense. And the same can be said of infinity as a would-be limit of infinitely large quantities.[9]

According to Portuondo, the idea of a variable quantity whose magnitude grows indefinitely embodies everything that can be conceived as infinity. He compares this view with that of ordinary space, understanding that *space is the abstraction of the place taken by a body, and thinking that the larger the body, the larger the space it needs, and indefinitely so.* In the same way, indefinitely small quantities give rise to the idea of zero as a time instant or of an isolated point in space.

In Chapter I, Portuondo establishes a classification and ordering of infinitesimal variables (Portuondo, 1880, p. 15). He starts by defining a *unit variable*, but he adds a new condition on it, thus separating from the views of Duhamel, Freycinet, or DuBois-Reymond:

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[6]Las variables que crecen indefinidamente reciben el nombre de variables **infinitamente grandes**, y las que decrecen indefinidamente son llamadas variables **infinitamente pequeñas**; aunque unas y otras serían más propiamente llamadas indefinidamente grandes e indefinidamente pequeñas. El Cálculo ó Análisis en que intervienen directamente esta clase de variables, es el Calculo o Análisis Infinitesimal.

[7]Por la definición de límite se ve que siempre que una variable tiende hacia un límite [no nulo], su diferencia con éste es una infinitamente pequeña; esta variable infinitamente pequeña *complementaria*, será función de las mismas independientes que la variable con límite, y ambas varían simultáneamente de tal modo que, cuando la una tiende hacia su límite, la otra, que es, por decirlo así, su *complemento*, decrece indefinidamente, ...

[8]A compactification of the real line, *e. g.* through a stereographic projection.

[9]Las variables, infinitamente pequeñas ó infinitamente grandes, *no tienen límite*. El *cero*, como cantidad constante que pueda ser límite de las variables infinitamente pequeñas, carece de sentido, y es igualmente ilusorio que el *infinito*, como constante que pueda ser límite de las variables infinitamente grandes.

> Let us consider all infinitely small variables, i.e. all conceivable laws of indefinite decrease: If we arbitrarily choose one of them -let us call it the unit variable- to use as a comparison standard, it will be needed that there exists a simultaneous correspondence between all laws, that is, for any state of the unit magnitude there exist well defined states for any other quantity.[10]

By fixing a unit variable, the order of an infinitely small quantity is established: Let u be the chosen unit, then any other infinitely small quantity v is of order n with respect to u if the limit of the ratio between v and the n-th power of u is a constant:

$$\lim \frac{\nu}{u^n} = k$$

and k is called *the value of the infinitely small law.* Therefore the following representation for v follows:

$$u^n(k + \omega)$$

where ω is the complementary variable of the quantity $\frac{\nu}{u^n}$.

Let us compare this definition with the one given by Cauchy. In the *Cours d'Analyse* (p. 28 ff.) an infinitesimal of order n is defined in terms of an infinitesimal unit u as

$$ku^n(1 \pm \varepsilon),$$

where ε is *"un nombre variable qui décroise indéfiniment avec la valeur numérique de u".* Therefore, Portuondo's definition is formally obtained from Cauchy's by writing

$$ku^n(1 \pm \varepsilon) = u^n(k \pm k\varepsilon) = u^n(k + \omega)$$

where the double sign $\pm$ is embodied in the complementary variable ω. The novelty in the approach by the Spanish author is to consider k as an important quantity in establishing the order of infinitesimal quantities, a fact that is not paid any attention to in the *Cours.*

Comparison of two infinitesimals $u^n(k + \omega)$ and $u^m(k + \omega')$ is made by computing the limit of their ratio. A table shows the results:

Orders:	$m > n$	$m = n$	$m < n$
Ratios:	$u^{m-n}(\frac{k}{h} + \omega_1)$	$\frac{k}{h}$	$u^{n-m}(\frac{h}{k} + \omega_2)$

Table 1: Comparing infinitely small quantities according to Portuondo.

A similar one can be written for infinitely large quantities in a footnote on page 28. It is also shown how to perform the change of the unit infinitesimal.

From the mathematical viewpoint these relationships establish an order of infinitesimal quantities that extends the usual order in the real line, although according to Borel and Hardy the so called DuBois-Reymond Theorem (DuBois-Reymond, 1875, p. 365), asserting that it is impossible to find a smallest element in the class of infinitesimally small quantities, is needed. Even though DuBois-Reymond's result was formulated several years before *Ensayo* was published, Portuondo seems not to be aware either of it or of its implications to the theory of infinitesimals. Our author resumes his views (Portuondo, 1880, p. 29) as follows (see Figure 6):

> We can say that once an arbitrarily chosen infinitesimal quantity is designed as unit, all infinitesimal laws (both infinitely small and infinitely large) are ordered in a continuous scale according to their infinitesimal

[10] Consideramos en una misma especie de cantidad todas las variables infinitamente pequeñas, es decir, todas las *leyes* imaginables de decrecimiento indefinido: si nos fijamos arbitrariamente en una de ellas (que llamaremos variable *unidad*) para que sirva como término de comparación, será preciso que haya correspondencia de simultaneidad entre todas las leyes; es decir, que a un determinado estado de magnitud de una variable, correspondan estados bien determinados de todas las demás.

order. Moreover, those with the same order are continuously ordered according to their values. In everyday language, we could say that the second scale is infinite at every point of the first infinite scale.[11]

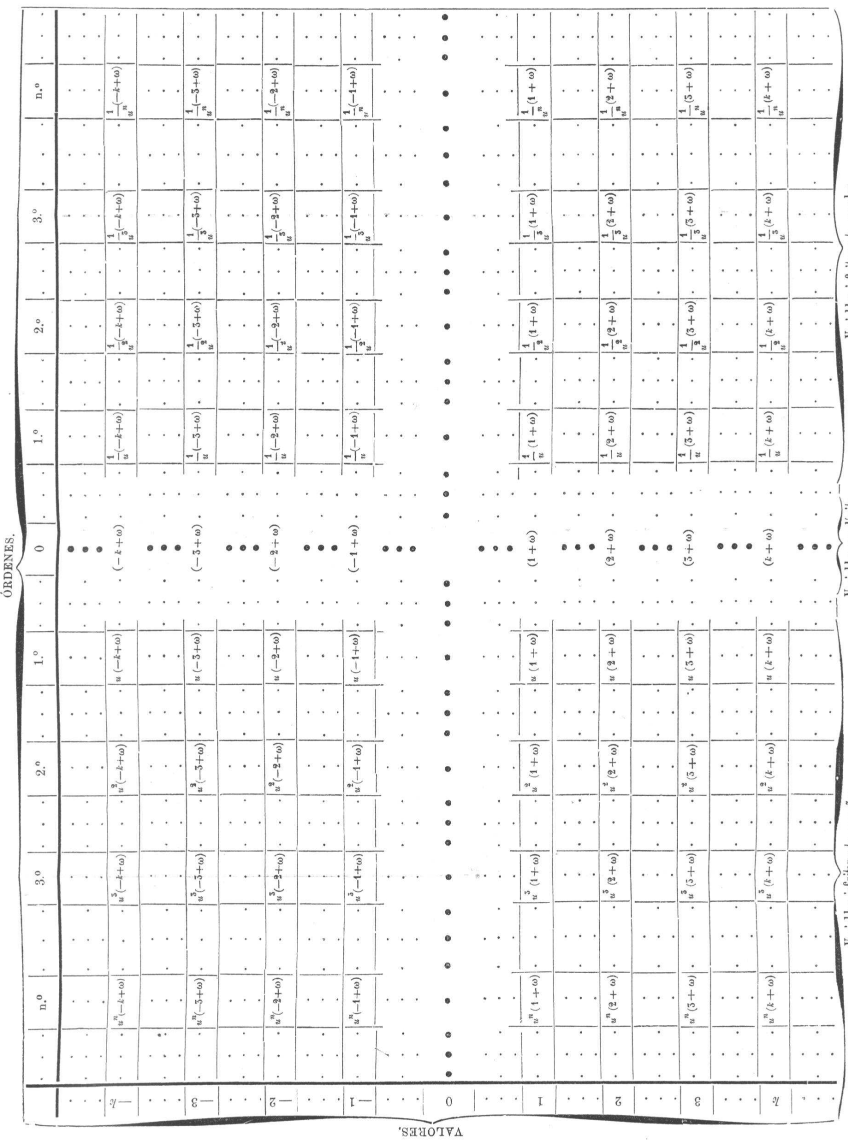

Figure 6: Ordering of infinitesimal quantities

The main difference between the perceptions of DuBois-Reymond and of the Spaniard is that this one considers a double ordering that takes into account the two characteristic parameters of infinitesimal quantities, while the German only considers one of them, the infinitesimal order. Therefore, according to Portuondo, if $\mu = u^m(k+\omega)$ and $\nu = u^n(l+\omega')$ are two infinitesimals, in order to assess that $\mu \succ \nu$, it must happen that $m \geq n$, and in case of equality the ordering criterion is $k \leq l$. To DuBois-Reymond, $\mu \preccurlyeq \nu$ if $m < n$. Moreover,

[11] Podemos decir que, escogida arbitrariamente una variable infinitamente pequeña y adoptada como *unidad*, todas las leyes infinitesimales (infinitamente pequeñas e infinitamente grandes) aparecen ordenadas en escala continua de *órdenes*, y las de un mismo orden colocadas en escala continua de *valores* ; hablando en lenguaje ordinario, se diría que esta última escala de valores es infinita en un punto de la primera escala infinita de los órdenes.

Portuondo points out that his procedure generates a *continuity law* between infinitesimal quantities similar to that of the real continuum. To show it, he starts with the same complementary variable ω for all infinitesimals. Thus, by letting k grow without bound in the expression $\nu = u^n(k+\omega)$, then k would upgrade to some other infinitesimal $u^{-p}(k_1+\omega)$ and therefore is obtained: an infinitesimal of order $n-p$, whose value k_1 is as small as desired,

$$\nu = u^n(u^{-p}(k_1+\omega)+\omega) = u^{n-p}(k_1+(\omega+u^{-p}\omega)),$$

and the result follows by observing that p can be arbitrarily small, as well as the difference between the values k and k_1. To conclude, it is shown that the condition that ω is the same for all infinitesimals can be dispensed with by remarking that a chain of infinitesimals can be written as $\omega = u^r(k'+\omega')$, $\omega' = u^r(k''+\omega'')$,..., and this sequence can be either an infinite one, or else end at some q for which $\omega^{q)} = k^q u^q$ and there is no complementary value. We must notice here that Portuondo is assuming without proof the Theorem of DuBois-Reymond.

Chapter II is devoted to the Algebra of infinitesimal quantities. Of course, addition, subtraction, etc. are thoroughly studied, both for infinitesimals and that for infinitesimals mixed with constant quantities The treatment is both analytic and geometrical, illustrated by the line drawings referred above (see Figure 7). The original contribution by Portuondo is that he elaborates the theory keeping in mind the two parameters, order and value, that define an infinitesimal once the unit has been chosen. Thus, given $\mu = u^m(k+\omega)$ and $\nu = u^n(h+\omega')$ the following results are obtained (see table 2):

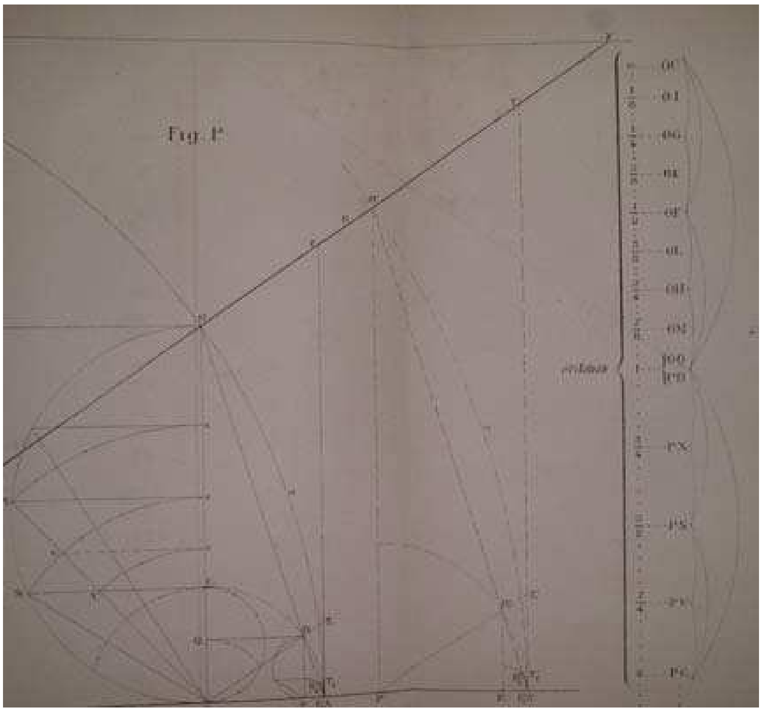

Figure 7. Fine line drawings related to chapter 2.

Orders:	$m \neq n$ $m > n$	$m = n$
Sum:	$u^n(h+(\omega'+u^{m-n}(k+\omega))) =$ $= u^n(h+\Omega)$	$u^n((h+k)+(\omega+\omega'))$

Table 2: On the sum of infinitesimal quantities.

In words, the sum has *the order and value of the summand with least order.* Of course, if the order is the same, the sum has the same order, and its value is the sum of the corresponding values. The results for subtraction are the same, except for the indeterminate case when they have the same values and orders. In this case the difference is an *infinitely small quantity with respect to any of the terms in the subtraction, and its order is therefore larger than the common order* of the original infinitesimals.

The theory carries on immediately to infinitely large quantities, for which the order is a negative number, $n < 0$, and Portuondo also studies the mixed case, when one of the quantities is an infinitely large and the other is an infinitely small, to establish that the sum or difference of two such quantities is always an infinitely large one with equal order and value than the given one.

The value and the order of the product are established through the general formula $\mu \cdot \nu = u^{m+n}(hk + \omega_1)$, and the quotient formula is $\frac{\mu}{\nu} = u^{m-n}(\frac{h}{k} + \omega_1)$, which is considered in more detail in the final *Nota* or appendix in the pages 126-142 of *Ensayo*.

Chapter III deals with explanations of the main theorems of Calculus. At the beginning, the principle of substitution of equivalent infinitesimals is stated, and as a start the author writes (Portuondo, 1880, p. 105-106):

> This theorem is the basis of *Differential Calculus*, because according to it, the nature of the complementary infinitesimals ω can be modified, in such a way that computations are freed from their most ordinary difficulties...[12]

The remaining pages are devoted to an analysis of quadratures, *i.e.* to the problem of an infinite sum of infinitely small quantities. Portuondo's argument is the following. Let all infinitely small quantities have the same order. Then their sum will be:

$$\begin{aligned} S &= u^m(k+\omega) + u^m(k'+\omega') + u^m(k''+\omega'') + \cdots = \\ &= u^m((k+k'+k''+\cdots) + (\omega+\omega'+\omega''+\cdots)) = u^m(K+\Omega) = \\ &= u^m\left(\frac{1}{u^n}(k_1+\omega_1)+\Omega\right) \end{aligned}$$

where $\Omega = \omega + \omega' + \omega'' + \cdots$ is always an infinitely small quantity. The results are summarized in the following table, where m = order of the infinitely small, n = order of the infinitely large K:

Order of the terms (m) and order of the sum (n)	$m > n$	$m = n$	$m < n$
Type of quadrature (Sum):	An infinitely small	Value of K	An infinitely large

Table 3: Types of definite integrals.

At this point Portuondo includes his only quotation (Portuondo, 1880, p. 109, footnote) of another author, with respect to the case $m = n$. It stems from the *Cours d'Analyse* (N° 16, Théoreme , p. 9, Vol. I) by Sturm:

> If a sum of infinitely small, whose number grows without bound, has a limit, then the sum of the products obtained by multiplying the infinitely small quantities times any other infinitely small will have zero limit.[13]

[12] Este teorema sirve de base al *Cálculo Diferencial*, porque se puede, en virtud de él, modificar el modo de ser de de las infinitamente pequeñas que aparecen en la relación, reemplazando éstas por las que más convengan (...) lo cual permite desembarazarse de lo que más dificulta ordinariamente la marcha de los Cálculos...

[13] Si une somme d'infiniment petits, dont le nombre augmente indéfiniment, a une limite finie, la somme des produits obtenus en les multipliant respectivement par d'autres infiniment petits aura pour limite zéro.

The body of the book closes with a short Conclusion where is treated, the idea of contact between two explicitly defined curves in the usual way: the contact is of order n at some point if the values of the functions and their first n derivatives are the same at the point. Therefore, from the Taylor expansions of both functions at the point, one obtain the following definition of an infinitesimal:

$$f(a+u) - \varphi(a+u) = u^n \left(\frac{f^{n+1}(a) - \varphi^{n+1}(a)}{(n+1)!} + \omega \right)$$

from which the author infers the usual rules for the contact of curves.

4.2 The final *Nota*, or appendix

This long note contains a study of the expression

$$\lim \frac{\varphi(a+u)}{f(a+u)} = \frac{\varphi^{(n)}(a)}{f^{(n)}(a)}$$

under the hypothesis that the functions

$$\varphi(a), \quad \varphi'(a), \quad \varphi''(a), \quad \varphi^{(n-1)}(a), \qquad f(a), \quad f'(a), \quad f''(a), \quad f^{(n-1)}(a)$$

are infinitesimals, either infinitely small or large. This endeavour, which amounts to thoroughly justifying L'Hôpital's rule, is very similar to that in (DuBois-Reymond, 1875). The main result is that the relationship between the values of the derivatives can be substituted for that between the values of the functions only in the case when the orders of $\varphi(a)$ and $f(a)$ are equal. The argument is rather involved and follows these lines (here, only for the infinitely small case):

Let $\varphi(a) = u^n \left(\frac{\varphi^{(n)}(a)}{n!} + \omega \right)$ and $f(a) = u^m \left(\frac{f^{(m)}(a)}{m!} + \omega' \right)$. Then their i-th derivatives satisfy $\varphi^{i)}(a) = u^{n-i} \left(\frac{\varphi^{(n)}(a)}{(n-i)!} + \omega \right)$ and $f^{i)}(a) = u^{m-i} \left(\frac{f^{(n)}(a)}{(m-i)!} + \omega \right)$, so after some algebra the following table is obtained:

Orders	**Expression for** $\frac{\varphi(a)}{f(a)}$
$m < n$	$u^{n-m} \left(\frac{\varphi^{n)}(a)}{f^{m)}(a)} \cdot \frac{1}{(m+1)(m+2)\cdots n} + \omega \right)$
$m = n$	$\frac{\varphi^{n)}(a)}{f^{n)}(a)} + \omega$
$n < m$	$u^{m-n} \left(\frac{\varphi^{n)}(a)}{f^{m)}(a)} (n+1)(n+2)\cdots m + \omega \right)$

Table 4: Comparison of quotients of functions and n-th derivatives.

5 Portuondo and Non-Standard Analysis

Infinitesimal theory was a forgotten item of Mathematics during the first half of the 20^{th} century, until it knew a second revival with the publication of *Non-standard Analysis* by Abraham Robinson (1918-1974) the year 1966. This book presents a theory for infinitesimals on a logical basis. According to Dauben (Dauben, 1998, p. 350), Robinson had that the dual problems of the status of infinities and infinitesimals were of more concern to mathematicians and philosophers than the problems in the foundations of Arithmetic discovered by Gödel. Dauben points out that, historically speaking, Arithmetic not been had a problem, while infinitesimals had always been involved in paradoxes and inconsistencies that made them subjects of more than one problematic situation. Robinson constructed an ordered set of *hyper-real numbers* where infinitesimal theories are fully consistent and can be applied to proving results on the real number domain (Robinson, 1966, p.2):

> It is shown in this book that Leibniz's ideas can be fully vindicated and that they lead to a novel and fruitful approach to classical Analysis and to many other branches of mathematics. The key to our method is provided by the detailed analysis of the relations between mathematical languages and mathematical structures which lies at the bottom of contemporary model theory.

In fact, Robinson acknowledges inspiration to Voltaire, whom he quotes in the preface (Robinson, 1966, p. VII):

> Je vois plus que jamais qu'il ne faut juger de rien sur sa grandeur apparente. O Dieu! qui avez donné une intelligence à des substances qui paraissent si méprisables, l'infiniment petit vous coûte aussi peu que l'infiniment grand. (Voltaire, *Micromegas ou voyages des habitants de l'étoile Sirius*, cap. VI).

We consider that *Ensayo* contains some aspects that could be related to Nonstandard Analysis. The set of all continuous with limit used by Portuondo could be identified with what we know today as a generalised sequence; therefore the starting point is the same as in Nonstandard Analysis. In *Ensayo* a classification into three disjoint classes is made according to whether the limit is zero, a finite nonzero number, or infinity.

According to Robinson, the quotient set obtained through the equivalence relationship "having infinitely many equal terms" is performed, and Portuondo's procedure can be shown to be equivalent to this one in terms of the value and the order of an infinitesimal. The basic operations are defined termwise in both cases. Nevertheless, in the ordering where both approaches diverge: The ordering of real numbers is continued into the infinitely small realm via the dominated convergence principle and, thus, a total ordering principle fully compatible with that of the real numbers is established. This approach recovers the Leibnizian idea of *actual infinitesimals*. The reader can see an interesting account in (Cuesta Dutari, 1980). Nevertheless, the ordering of Portuondo, considering order and value, is not an extension of the natural order of the real numbers, as can be easily proved.

On the other hand, the ideas of derivative and definite integral are found to be similar in our author and in the Nonstandard Analysis version.

6 Conclusions

Portuondo's *Ensayo* aimed to establish a sound foundation to the arithmetic and order properties of infinitesimal quantities. As an isolated attempt, it deserves a place in the history of Spanish Mathematics. The standard of rigour in *Ensayo* is rather high, and it contains a very interesting attempt to spread light on various obscure points in the imported texts -especially French ones- that were common fare in Spain at the time.

The justification of the rule of L'Hôpital is thoroughly explained and can be considered, with the analysis of definite integrals, the main highlights in the book. An study of the article by Fisher (Fisher, 1978) shows that the contribution of Portuondo is in line with that of other authors of his time on the same topic.

On the relationship between this approach to infinitesimals and Nonstandard Analysis, the choice of an order where the actual value of the limit is an important element, though useful in computations where one looks closely at tangible results, is an non-necessary complication that really does not help much in the future developments of the theory.

There are some points open for further research, such as establishing the scholarly background of Antonio Portuondo and the reasons of his long silences of thirty two and forty seven years on the topic of infinitesimals, as well as his connection with mathematical methods applied to Social Sciences.

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